Analytical methods for Lévy processes with applications to finance, Part II

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Joint work with Alexey Kuznetsov.
1 Introduction

2 Asian options and meromorphic Lévy processes
   - Theory
   - Numerics and Implementation

3 Approximating Lévy processes with completely monotone jumps
Overview

- Calculate the price of an Asian option when the stock price is driven by a meromorphic process.
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  - Determine the Mellin transform and subsequently the distribution of $I_{e(q)}$. (theory)
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Numerical results

- Calculate the price of a barrier option when the stock price is any of the processes discussed thus far.

Classify our processes as processes with completely monotone jump densities

- Demonstrate how easily approximate any completely monotone process by a hyper-exponential process

Price barrier options (or asian options, or lookback options,...) using the hyper-exponential process and the Wiener-Hopf factorization.

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Other pricing methods

In general, pricing Asian options is difficult because they are path dependent options and $Z_t = A_0 \int_0^t e^{X_u} du$ is not a Markov process.

1. Monte Carlo simulation

2. Moment matching, Black-Scholes setting


3. Reducing to a PDE or IDE and solving numerically:
   - The two-dimensional process $(X_t, Z_t)$ is Markov. Derive three-dimensional PDE for $C$.

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1. Monte Carlo simulation
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3. Reducing to a PDE or IDE and solving numerically:
   - The two-dimensional process \((X_t, Z_t)\) is Markov. Derive three-dimensional PDE for \(C\).
   - Write \(C\) in terms of \(\tilde{Z}_t := (x + Z_t)e^{-X_t}\) by a change of measure. Since \(\tilde{Z}_t\) is Markov, we can compute \(C\) by solving the backward Kolmogorov equation (two-dimensional IDE).
The distribution of $I_{e(q)}$

- The hyper-exponential case (finite activity jumps)
  

- Processes with jumps of rational transform (finite activity jumps)
  

- Hyper-geometric processes (infinite activity jumps but distribution is known for only one value of $q$)
  
Asian call

Recall, we wish to compute

\[ C(A_0, K, T) := e^{-rT} \mathbb{E} \left[ \left( \frac{1}{T} \int_0^T A_0 e^{X_u} du - K \right)^+ \right], \]

or equivalently compute

\[ f(k, t) := \mathbb{E} \left[ \left( \int_0^t e^{X_u} du - k \right)^+ \right]. \]
Asian call

Our proposed algorithm follows Cai and Kou. That is, we transform once

\[ h(k, q) := q \int_{\mathbb{R}^+} e^{-qt} f(k, t) dt = \mathbb{E} \left[ \left( \int_{0}^{\mathcal{E}(q)} e^{X_t} dt - k \right)^+ \right], \]

and then again

\[ \Phi(z, q) := \int_{\mathbb{R}^+} h(k, q) k^{z-1} dk = \mathbb{E} \left[ \int_{\mathbb{R}^+} (I_{\mathcal{E}(q)} - k)^+ k^{z-1} dk \right] \]

\[ = \mathbb{E} \left[ \int_{0}^{I_{\mathcal{E}(q)}} (I_{\mathcal{E}(q)} - k) k^{z-1} dk \right] = \frac{\mathbb{E} \left[ I_{\mathcal{E}(q)}^{z+1} \right]}{z(z + 1)} = \frac{\mathcal{M}(I_{\mathcal{E}(q)}, z + 2)}{z(z + 1)}, \]

to get an expression for the doubly transform price in terms of the Mellin transform of the exponential functional.
Products of Beta random variables

With any two unbounded sequences $\alpha = \{\alpha_n\}_{n \geq 1}$ and $\beta = \{\beta_n\}_{n \geq 1}$ which satisfy the interlacing property

$$0 < \alpha_1 < \beta_1 < \alpha_2 < \beta_2 < \alpha_3 < \beta_3 \ldots$$

we associate an infinite product of independent beta random variables, defined as

$$J(\alpha, \beta) := \prod_{n \geq 1} B(\alpha_n, \beta_n - \alpha_n) \frac{\beta_n}{\alpha_n}.$$

Lemma

$J(\alpha, \beta)$ converges a.s.
Main Result

Theorem (H. and Kuznetsov, 2014)

Assume that $q > 0$. Define $\hat{\rho}_0 := 0$ and the four sequences

$\zeta := \{\zeta_n\}_{n \geq 1}$, $\rho := \{\rho_n\}_{n \geq 1}$, $\tilde{\zeta} := \{1 + \zeta_n\}_{n \geq 1}$, $\tilde{\rho} := \{1 + \rho_{n-1}\}_{n \geq 1}$.

Then we have the following identity in distribution

$$I_{e(q)} \overset{d}{=} C(q) \times \frac{J(\tilde{\rho}, \tilde{\zeta})}{J(\zeta, \rho)},$$

where $C(q)$ is a constant and the random variables $J(\tilde{\rho}, \tilde{\zeta})$ and $J(\zeta, \rho)$ are independent. cont. $\rightarrow$
Main Result

Theorem (cont.)

The Mellin transform $\mathcal{M}(I_{e(q)}, z)$ is finite for $0 < \text{Re}(z) < 1 + \zeta_1$ and is given by

$$
\mathcal{M}(I_{e(q)}, z) = C^{z-1} \prod_{n \geq 1} \frac{\Gamma(\hat{\zeta}_n + 1) \Gamma(\hat{\rho}_n - 1 + z)}{\Gamma(\hat{\rho}_n - 1 + 1) \Gamma(\hat{\zeta}_n + z)} \left( \frac{\hat{\zeta}_n + 1}{\hat{\rho}_n - 1 + 1} \right)^{z-1} \times
$$

$$
\prod_{n \geq 1} \frac{\Gamma(\rho_n) \Gamma(\zeta_n + 1 - z)}{\Gamma(\zeta_n) \Gamma(\rho_n + 1 - z)} \left( \frac{\zeta_n}{\rho_n} \right)^{z-1} \mathcal{M}(J(\zeta, \rho), 2-z).
$$

D. Hackmann and A. Kuznetsov.
Asian options and meromorphic Lévy processes.
A rough idea of the proof

We use the verification result of Kuznetsov and Pardo: A function $f(z)$ is the Mellin transform of $I_{e(q)}$ if

1. for some $\theta > 0$, the function $f(z)$ is analytic and zero free in the vertical strip $0 < \text{Re}(z) < 1 + \theta$;
2. the function $f(z)$ satisfies
   \[ f(z + 1) = \frac{z}{q - \psi(z)} f(z), \quad 0 < z < \theta, \]
   where $\psi(z)$ is the Laplace exponent of the process $X$;
3. $|f(z)|^{-1} = o(\exp(2\pi|\text{Im}(z)|))$ as $\text{Im}(z) \to \infty$, uniformly in the strip $0 < \text{Re}(z) < 1 + \theta$. 
A rough idea of the proof

We need to find a candidate function \( f(z) \) and we let point 2 guide us. We are aided by the fact that \( q - \psi(z) \) is just a product of linear factors involving the roots and poles. That is,

\[
q - \psi(z) = q \prod_{n \geq 1} \frac{1 - \frac{z}{\zeta_n}}{1 - \frac{z}{\rho_n}} \times \prod_{n \geq 1} \frac{1 + \frac{z}{\zeta_n}}{1 + \frac{z}{\rho_n}}, \quad z \in \mathbb{C},
\]

where the two infinite products converge.

A rough idea of the proof

Therefore, we are solving many simpler functional equations of the form:

\[ f(z + 1) = (a \pm z)^k f(z), \]

where \( a \) represents a root or a pole, and \( k \in \{-1, 1\} \). A solution of such an equation can readily be obtained using the well known formula

\[ \Gamma(z + 1) = z\Gamma(z), \]

for the gamma function.
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To obtain the price we need to compute $h(k, q)$ as the inverse Mellin transform

$$h(k, q) = \frac{k^{-d_1}}{2\pi} \int_{\mathbb{R}} \frac{\mathcal{M}(I_{e(q)}, d_1 + iv + 2)}{(d_1 + iv)(d_1 + iv + 1)} e^{-iv \log(k)} dv,$$

where $d_1 \in (0, \zeta_1(d_2) - 1)$, $q = d_2 + iu$, and $d_2 > r$. Second, we compute $f(k, t)$ as the inverse Laplace transform, which can be rewritten as the cosine transform

$$f(k, t) = \frac{2e^{d_2 t}}{\pi} \int_{\mathbb{R}^+} \text{Re} \left( \frac{h(k, d_2 + iu)}{d_2 + iu} \right) \cos(ut) du.$$
Implementation

The steps we need to follow/hurdles we need to overcome are:

- Choose a process
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- Choose a process
- Evaluate $\mathcal{M}(I_{e(q)}, z)$ for complex $q$
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- Choose a process
- Evaluate $\mathcal{M}(I_{e(q)}, z)$ for complex $q$
- Truncate $\mathcal{M}(I_{e(q)}, z)$ efficiently
The process

We will use a theta process for which we have a closed form formula for $\psi(z)$. We can manipulate parameters of the the process to give a process with infinite activity and variation.

Parameter Set I will give a process with a Gaussian component and jumps of infinite activity but finite variation.

Parameter Set II gives a process with zero Gaussian component and jumps of infinite variation.
Complex $q$

Unfortunately, we do not know whether or formula for $\mathcal{M}(I_{e(q)}, z)$ is valid for complex $q$. Our numerical experiments support the conjecture that it is.

What about finding the roots $\{\zeta_n, -\hat{\zeta}_n\}_{n \geq 1}$ when $q = q_0 + iu$, $u \in \mathbb{R}^+$?
Complex $q$

We may view $\zeta_n(u)$ as an implicitly defined function of $u$ which satisfies,

$$q_0 + iu - \psi(\zeta_n(u)) = 0, \quad \zeta_n(0) = \zeta_n,$$

where $\zeta_n$ is the solution of $\psi(z) = q_0$. Differentiating each side with respect to $u$ gives the ordinary differential equation

$$\frac{d}{du}\zeta_n(u) = \frac{i}{\psi'(\zeta_n(u))},$$

with initial condition $\zeta_n(0) = \zeta_n$. Such an equation can be solved nicely by a numerical scheme like the midpoint method.
To approximate $\mathcal{M}(I_{e(q)}, z)$ we can simply truncate our infinite product, but convergence may be slow. The more terms we need, the more roots $\{\zeta_n, \hat{\zeta}_n\}_{n \geq 1}$ we need to calculate which is computationally expensive. Note if we truncate the transform we get:

$$\mathcal{M}_N(z) := a_N \times b_N^{z-1} \times \prod_{n=1}^{N} \frac{\Gamma(\hat{\rho}_n - 1 + z)}{\Gamma(\hat{\zeta}_n + z)} \frac{\Gamma(\zeta_n + 1 - z)}{\Gamma(\rho_n + 1 - z)}$$

where and $a_N$ and $b_N$ are normalizing constants.
Truncating $\mathcal{M}(I_{e(q)}, z)$

Now we note that

$$\mathcal{M}(I_{e(q)}, z) = \mathcal{M}_N(z)R_N(z)$$

where $R_N(z) = \mathcal{M}(I_{e(q)}, z)/\mathcal{M}_N(z)$ is the Mellin transform of the tail of our product of beta random variables which we denote $\epsilon^{(N)}$. Instead of simply letting $R_N(s) = 1$ we try to find a random variable $\xi$ matching the moments of $\epsilon^{(N)}$. 
Truncating $\mathcal{M}(I_{e(q)}, z)$

We can calculate the moments $m_k$ using the functional equation

$$\mathcal{M}(I_{e(q)}, z + 1) = z\mathcal{M}(I_{e(q)}, z)/(q - \psi(z)),$$

we find

$$m_k = R_N(k + 1) = \frac{\mathcal{M}(I_{e(q)}, k + 1)}{\mathcal{M}_N(k + 1)} = \frac{k!}{\mathcal{M}_N(k + 1)} \prod_{j=1}^{k} \frac{1}{q - \psi(j)}.$$
Truncating $M(I_{e(q)}, z)$

Finally we let $\xi$ be a beta random variable of the second kind which has density:

$$
P(\xi \in dx) = \frac{\Gamma(a)\Gamma(b)}{\Gamma(a + b)} y^{a-1}(1 + y)^{-a-b} dy, \quad y > 0.
$$

We choose $a, b > 0$ such $\mathbb{E}[\xi] = m_1$ and $\mathbb{E}[\xi^2] = m_2$, and replace $R_N(z)$ with the Mellin transform of $\xi$ which has the form:

$$
\mathbb{E}[\xi^{z-1}] = \frac{\Gamma(a + z - 1)\Gamma(b + 1 - z)}{\Gamma(a)\Gamma(b)}.
$$
A test: Calculating the density of $I_{e(1)}$

Figure: (a) The density of the exponential functional $I_{e(1)}$ with $N = 400$ (the benchmark). (b) The error with $N = 20$ (no correction). (c) The error with $N = 20$ (with correction term). Solid line (resp. circles) represent parameter set I (resp. II).
Numerics: Pricing an Asian Option Results

<table>
<thead>
<tr>
<th>$N$</th>
<th>Algorithm 1, price</th>
<th>Time (sec.)</th>
<th>Algorithm 2, price</th>
<th>Time (sec.)</th>
</tr>
</thead>
<tbody>
<tr>
<td>10</td>
<td>4.724627</td>
<td>1.6</td>
<td>4.720675</td>
<td>1.2</td>
</tr>
<tr>
<td>20</td>
<td>4.727780</td>
<td>2.8</td>
<td>4.728032</td>
<td>1.8</td>
</tr>
<tr>
<td>40</td>
<td>4.728013</td>
<td>4.8</td>
<td>4.728031</td>
<td>3.4</td>
</tr>
<tr>
<td>80</td>
<td>4.728029</td>
<td>9.2</td>
<td>4.728031</td>
<td>7.1</td>
</tr>
</tbody>
</table>

Table: The price of the Asian option, parameter set I. The Monte-Carlo estimate of the price is 4.7386 with the standard deviation 0.0172. The exact price is 4.72802$\pm$1.0e-5.

Option parameters: $A_0 = 100$, $T = 1$, $K = 105$, and $r = 0.03$, with risk neutral condition $\psi(1) = r$ satisfied (this and the assumption $\rho_1 > 1$ ensures key quantities are finite).
Numerics: Pricing an Asian Option Results

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<th>$N$</th>
<th>Algorithm 1, price</th>
<th>Time (sec.)</th>
<th>Algorithm 2, price</th>
<th>Time (sec.)</th>
</tr>
</thead>
<tbody>
<tr>
<td>10</td>
<td>10.620243</td>
<td>1.6</td>
<td>10.621039</td>
<td>1.2</td>
</tr>
<tr>
<td>20</td>
<td>10.620049</td>
<td>3.0</td>
<td>10.620171</td>
<td>2.2</td>
</tr>
<tr>
<td>40</td>
<td>10.620037</td>
<td>4.8</td>
<td>10.620054</td>
<td>3.6</td>
</tr>
<tr>
<td>80</td>
<td>10.620036</td>
<td>9.6</td>
<td>10.620039</td>
<td>7.4</td>
</tr>
</tbody>
</table>

Table: The price of the Asian option, parameter set II. The Monte-Carlo estimate of the price is 10.6136 with the standard deviation 0.0251. The exact price is 10.62003±1.0e-5.
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Classification: Completely monotone jumps

Definition

A function $f(x)$ is called completely monotone if $(-1)^n f^{(n)}(x) > 0$ for all $x > 0$, $n = 0, 1, 2, \ldots$.

Definition

A Lévy process has completely monotone jumps, if the Lévy measure is absolutely continuous with density $\pi(x)$, and $\pi(x)$ and $\pi(-x)$ are completely monotone for $x \in (0, \infty)$.

Assumption: From now on we assume all processes have completely monotone jumps and $\pi(x)$ decreases exponentially fast as $x \to \pm \infty$. 
Some facts

- All of the processes mentioned satisfy our assumption.
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- Hyper-exponential processes are dense in the class of completely monotone processes in the sense of weak convergence.
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M. Jeannin and M. Pistorius.
A transform approach to compute prices and Greeks of barrier options driven by a class of Lévy processes.
*Quantitative Finance, 10:629–644, 2010.*
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- The jump density of a process $X$ is completely monotone if, and only if, $S_q$ and $I_q$ are mixtures of exponentials.
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L.C.G. Rogers.
Weiner-Hopf factorization of diffusions and Lévy processes.
Main idea

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- Approximating a Lévy process is equivalent to approximating its Laplace exponent $\psi(z)$.
- The Laplace exponent of a hyper-exponential process is a rational function.
- Thus we have two problems:
  1. Approximate $\psi(z)$ by a rational function $\tilde{\psi}(z)$,
  2. Show that $\tilde{\psi}(z)$ is itself a Laplace exponent of a Lévy process.
Padé approximation

Definition

Let $f$ be a function with a power series representation $f(z) = \sum_{i=0}^{\infty} c_i z^i$. If there exist polynomials $P_m(z)$ and $Q_n(z)$ satisfying $\deg(P) \leq m$, $\deg(Q) \leq n$, $Q_n(0) = 1$ and

$$\frac{P_m(z)}{Q_n(z)} = c_0 + c_1 z + \cdots + c_{m+n} z^{m+n} + O(z^{m+n+1}), \quad z \to 0,$$

then we say that $f^{[m/n]}(z) := P_m(z)/Q_n(z)$ is the $[m/n]$ Padé approximant of $f$. 

Analytical methods

Daniel Hackmann
### A simple example of Padé approximations

<table>
<thead>
<tr>
<th>$m \backslash n$</th>
<th>0</th>
<th>1</th>
<th>2</th>
<th>3</th>
</tr>
</thead>
<tbody>
<tr>
<td>0</td>
<td>$\frac{1}{1}$</td>
<td>$\frac{1}{1-z}$</td>
<td>$\frac{1}{1-z+\frac{1}{6}z^2}$</td>
<td>$\frac{1}{1-z+\frac{1}{6}z^2-\frac{1}{30}z^3}$</td>
</tr>
<tr>
<td>1</td>
<td>$\frac{1+z}{1}$</td>
<td>$\frac{1+z}{1-z}$</td>
<td>$\frac{1+z}{1-z+\frac{1}{6}z^2}$</td>
<td>$\frac{1+z}{1-z+\frac{1}{6}z^2-\frac{1}{30}z^3}$</td>
</tr>
<tr>
<td>2</td>
<td>$\frac{1+z+\frac{1}{6}z^2}{1}$</td>
<td>$\frac{1+z+\frac{2}{3}z^2}{1-z}$</td>
<td>$\frac{1+z+\frac{2}{3}z^2}{1-z+\frac{1}{6}z^2}$</td>
<td>$\frac{1+z+\frac{2}{3}z^2}{1-z+\frac{1}{6}z^2-\frac{1}{30}z^3}$</td>
</tr>
<tr>
<td>3</td>
<td>$\frac{1+z+\frac{1}{6}z^2+\frac{1}{51}z^3}{1}$</td>
<td>$\frac{1+z+\frac{3}{4}z^2+\frac{1}{24}z^3}{1-z}$</td>
<td>$\frac{1+z+\frac{3}{4}z^2+\frac{1}{24}z^3}{1-z+\frac{1}{6}z^2}$</td>
<td>$\frac{1+z+\frac{3}{4}z^2+\frac{1}{24}z^3}{1-z+\frac{1}{6}z^2-\frac{1}{30}z^3}$</td>
</tr>
<tr>
<td>4</td>
<td>$\frac{1+z+\frac{1}{6}z^2+\frac{1}{51}z^3+\frac{1}{24}z^4}{1}$</td>
<td>$\frac{1+z+\frac{3}{4}z^2+\frac{1}{15}z^3+\frac{1}{120}z^4}{1-z}$</td>
<td>$\frac{1+z+\frac{3}{4}z^2+\frac{1}{15}z^3+\frac{1}{120}z^4}{1-z+\frac{1}{6}z^2}$</td>
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</tr>
</tbody>
</table>

**Figure:** The initial part of the Padé table for $e^z$
For each $n$ we want to find a measure $\tilde{\nu}_n$ on a finite number of points in $[a, b]$ such that we match the first $2^n - 1$ moments of $\nu$, i.e. 
\[ \int_{a}^{b} x^j \nu(dx) = \sum_{i=1}^{2^n-1} x^j w_i, \]
for $j = 1, \ldots, 2^n - 1$.

The points $x_i$ and $w_i$ are the nodes and weights of the Gaussian quadrature.

- $\nu$ is a finite positive measure on a closed bounded interval $[a, b]$
Gaussian quadrature

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Gaussian quadrature

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- For each $n$ we want to find a measure $\tilde{\nu}_n$ on a finite number of points in $[a, b]$ such that we match the first $2n - 1$ moments of $\nu$, i.e.

$$\int_{[a, b]} x^j \nu(dx) = \sum_{i} x_i^j w_i, \text{ for } j = 1, \ldots, 2n - 1.$$

- The points $x_i$ and $w_i$ are the nodes and weights of the Gaussian quadrature.
Gaussian quadrature and orthogonal polynomials

- \{p_n(x)\}_{n \geq 0} \text{ be the sequence of orthogonal polynomials with respect to the measure } \nu(dx): \deg(p_n) = n \text{ and }

\[(p_n, p_m)_\nu := \int_{[a,b]} p_n(x)p_m(x)\nu(dx) = d_n\delta_{n,m}\]

G. Szegö.
Orthogonal Polynomials.
Gaussian quadrature and orthogonal polynomials

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\[
(p_n, p_m)_\nu := \int_{[a,b]} p_n(x)p_m(x)\nu(dx) = d_n \delta_{n,m}
\]

- The nodes of the Gaussian quadrature \( \tilde{\nu}_n \) are the zeros of \( p_n \) and the weights may be calculated from \( p_{n-1}, p_n \).

G. Szegö.  
*Orthogonal Polynomials.*  
We can develop a very useful description of the processes which satisfy our assumption using Bernstein’s theorem. A process satisfies our assumption if, and only if, there exists a positive measure $\mu(du)$, with support in $\mathbb{R}\{0\}$, such that for all $x \in \mathbb{R}$

$$\pi(x) = \mathbb{I}(x > 0) \int_{(0,\infty)} e^{-ux} \mu(du) + \mathbb{I}(x < 0) \int_{(-\infty,0)} e^{-ux} \mu(du), \quad (1)$$

and $\mu(du)$ assigns no mass to a non-empty interval $(-\hat{\rho}, \rho)$ containing the origin + integrability condition on $\mu(du)$. 
A change of variables

We define

$$\mu^*(A) := \mu(\{v \in \mathbb{R} : v^{-1} \in A\}).$$

Then, the Lévy-Khintchine formula + Fubini’s theorem + change of variables give us

$$\psi(z) = \frac{\sigma^2}{2} z^2 + az + z^2 \int_{[-\hat{\rho}^{-1}, \rho^{-1}]} \frac{|v|^3 \mu^*(dv)}{1 - vz}.$$

**Key Observation:** $|v|^3 \mu^*(dv)$ is a finite measure, with bounded support.
Main theorem (two-sided case)

Assume that $\sigma = 0$. Let $\{x_i\}_{1 \leq i \leq n}$ and $\{w_i\}_{1 \leq i \leq n}$ be the nodes and the weights of the Gaussian quadrature of order $n$ with respect to the measure $|v|^3 \mu^*(dv)$. We define

$$\psi_n(z) := az + z^2 \sum_{i=1}^{n} \frac{w_i}{1 - zx_i}.$$  

Theorem (H. and Kuznetsov, 2014)

(i) The function $\psi_n(z)$ is the $[n + 1/n]$ Padé approximant of $\psi(z)$.

(ii) The function $\psi_n(z)$ is the Laplace exponent of a hyper-exponential process $X^{(n)}$ having the characteristic triple $(a, \sigma^2_n, \pi_n)_{h \equiv x}$, where
Main theorem (two-sided case)

Theorem (cont.)

(ii) \( \pi_n(x) := \begin{cases} \sum_{1 \leq i \leq n : x_i < 0} w_i |x_i|^{-3} e^{-\frac{x}{x_i}}, & \text{if } x < 0, \\ \sum_{1 \leq i \leq n : x_i > 0} w_i x_i^{-3} e^{-\frac{x}{x_i}}, & \text{if } x > 0. \end{cases} \)

(iii) The random variables \( X_1^{(n)} \) and \( X_1 \) satisfy \( \mathbb{E}[(X_1^{(n)})^j] = \mathbb{E}[(X_1)^j] \) for \( 1 \leq j \leq 2n + 1 \).
Theorem (H. and Kuznetso, 2014)

For any compact set \( A \subset \mathbb{C} \setminus \{(-\infty, -\hat{\rho}] \cup [\rho, \infty)\} \) there exist \( c_1 = c_1(A) > 0 \) and \( c_2 = c_2(A) > 0 \) such that for all \( z \in A \) and all \( n \geq 1 \)

\[
|\psi_n(z) - \psi(z)| < c_1 e^{-c_2 n}.
\]

D. Hackmann and A. Kuznetso.
Approximating Lévy processes with completely monotone jumps.
One-sided processes

- For CM subordinators, all three functions $\psi[n/n](z)$, $\psi[n+1/n](z)$, $\psi[n+2/n](z)$ are Laplace exponents of hyper-exponential processes.
One-sided processes

- For CM subordinators, all three functions $\psi_{n/n}(z)$, $\psi_{n+1/n}(z)$, $\psi_{n+2/n}(z)$ are Laplace exponents of hyper-exponential processes.
- For CM spectrally-positive processes of infinite variation, only two functions $\psi_{n+1/n}(z)$, $\psi_{n+2/n}(z)$ are Laplace exponents of hyper-exponential processes.
One-sided processes

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- For CM spectrally-positive processes of infinite variation, only two functions $\psi^{[n+1/n]}(z)$, $\psi^{[n+2/n]}(z)$ are Laplace exponents of hyper-exponential processes.

- There exist explicit formulas for a number of important examples:
  In the VG case we have $\psi^{[n/n]}(z) = P_n(z)/Q_n(z)$, where

  $$\begin{align*}
P_n(z) &= 2 \sum_{j=0}^{n} \binom{n}{j}^2 [H_{n-j} - H_j] (1 - z)^j, \\
Q_n(z) &= z^n P_n \left( \frac{2}{z} - 1 \right).
\end{align*}$$

  and $H_j := 1 + 1/2 + \cdots + 1/j$. 
How do we prove all these results?

- One can show that only \( \psi[n/n](z) \), \( \psi[n+1/n](z) \) and \( \psi[n+2/n](z) \) can possibly be Laplace exponents of a Lévy process.
How do we prove all these results?

- One can show that only $\psi^{[n/n]}(z)$, $\psi^{[n+1/n]}(z)$ and $\psi^{[n+2/n]}(z)$ can possibly be Laplace exponents of a Lévy process.

- The function

$$g(z) = \int_{[-\hat{\rho}^{-1},\rho^{-1}]} \frac{|v|^3 \mu^*(dv)}{1 - vz}.$$  

is closely related to a Stieltjes function:

$$f(z) := \int_{[0,R^{-1}]} \frac{\nu(du)}{1 + zu}.$$
Some more theory on Stieltjes functions.

- $f^{[m/n]}(z)$ always exists provided $m \geq n - 1$. 


Some more theory on Stieltjes functions.

- $f^{[m/n]}(z)$ always exists provided $m \geq n - 1$.
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$$
\begin{align*}
  f^{[n-1/n]}(z) &= \frac{(-z)^{n-1}q_{n-1}(-1/z)}{(-z)^np_{n}(-1/z)} = \sum_{i=1}^{n} \frac{w_i}{1 + x_i z}.
\end{align*}
$$
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- Plus convergence results

  
  *Padé Approximants.*  
  

  
  Padé approximation of Stieltjes series.  
  
Math Finance applications

We will work with the following two processes: the VG process $V$ defined by the Laplace exponent

$$
\psi(z) = \mu z - c \log \left( 1 - \frac{z}{\rho} \right) - c \log \left( 1 + \frac{z}{\hat{\rho}} \right),
$$

and parameters

$$(\rho, \hat{\rho}, c) = (21.8735, 56.4414, 5.0),$$

and the CGMY process $Z$ defined by the Laplace exponent

$$
\psi(z) = \mu z + CT \Gamma(-Y) \left[ (M - z)^Y - M^Y + (G + z)^Y - G^Y \right],
$$

and parameters

$$(C, G, M, Y) = (1, 8.8, 14.5, 1.2).$$
A test: European call

<table>
<thead>
<tr>
<th></th>
<th>two-sided ([2N + 1/2N])</th>
<th>one-sided ([N + 1/N])</th>
<th>one-sided ([N + 2/N])</th>
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</thead>
<tbody>
<tr>
<td>(N = 1)</td>
<td>-2.75e-2</td>
<td>1.93e-2</td>
<td>-3.72e-3</td>
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<tr>
<td>(N = 2)</td>
<td>-4.86e-6</td>
<td>-4.19e-6</td>
<td>9.5e-5</td>
</tr>
<tr>
<td>(N = 3)</td>
<td>4.80e-7</td>
<td>-1.48e-5</td>
<td>-2.54e-7</td>
</tr>
<tr>
<td>(N = 4)</td>
<td>2.9e-8</td>
<td>6.41e-7</td>
<td>-1.55e-7</td>
</tr>
<tr>
<td>(N = 5)</td>
<td>1.14e-9</td>
<td>5.58e-9</td>
<td>6.95e-9</td>
</tr>
</tbody>
</table>

**Table**: The error in computing the price of the European call option for the CGMY Z-model. Initial stock price is \(A_0 = 100\), strike price \(K = 100\), maturity \(T = 0.25\) and interest rate \(r = 0.04\). The benchmark price is 11.9207826467.
Down-and-out put

\[ p_I(y) = \sum_{i=1}^{\hat{N}+1} \hat{c}_i \hat{\zeta}_i e^{\hat{\zeta}_i y}, \quad y < 0, \quad \text{and} \quad p_S(x) = \sum_{j=1}^{N+1} c_j \zeta_j e^{-\zeta_j x}, \quad x > 0. \]

\[ F(q) = \mathbb{E}[(k - e^{S_q+I_q})^+ \mathbb{I}(I_q > b)] \]

\[ = \int_{-b}^{0} \int_{\log(k)+y}^{0} (k - e^{x-y})p_S(x)p_I(-y)dx dy \]

\[ = \sum_{i=1}^{\hat{N}+1} \sum_{j=1}^{N+1} \frac{\hat{c}_i c_j}{\zeta_j - 1} \times \]

\[ \left(k(e^{b\hat{\zeta}_i} - 1)(1 - \zeta_j) - \frac{k^{1-\zeta_j} \hat{\zeta}_i(e^{b(\zeta_j+\hat{\zeta}_i)} - 1)}{\hat{\zeta}_i + \zeta_j} + \frac{\hat{\zeta}_i \zeta_j(e^{b(1+\hat{\zeta}_i)} - 1)}{\hat{\zeta}_i + 1} \right). \]
Down-and-out put

We calculate barrier option prices for the process $V$, for four values $A_0 \in \{81, 91, 101, 111\}$ and with other parameters given by $K = 100$, $B = 80$, $r = 0.04879$ and $T = 0.5$

<table>
<thead>
<tr>
<th></th>
<th>$A_0 = 81$</th>
<th>$A_0 = 91$</th>
<th>$A_0 = 101$</th>
<th>$A_0 = 111$</th>
</tr>
</thead>
<tbody>
<tr>
<td>Benchmark</td>
<td>3.39880</td>
<td>7.38668</td>
<td>1.40351</td>
<td>0.04280</td>
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<tr>
<td>$N = 2$</td>
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<td>7.39225</td>
<td>1.40527</td>
<td>0.04233</td>
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<td>$N = 4$</td>
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<td>7.38957</td>
<td>1.40329</td>
<td>0.04258</td>
</tr>
<tr>
<td>$N = 6$</td>
<td>3.39910</td>
<td>7.38939</td>
<td>1.40332</td>
<td>0.04258</td>
</tr>
<tr>
<td>$N = 8$</td>
<td>3.39856</td>
<td>7.38936</td>
<td>1.40332</td>
<td>0.04258</td>
</tr>
<tr>
<td>$N = 10$</td>
<td>3.39853</td>
<td>7.38936</td>
<td>1.40332</td>
<td>0.04258</td>
</tr>
</tbody>
</table>

**Table**: Barrier option prices calculated for the VG process $V$-model. Benchmark prices obtained from Kudryavtsev and Levendorskiĭ 2009

O. Kudryavtsev and S. Levendorskiĭ.

Fast and accurate pricing of barrier options under Lévy processes.