Karhunen-Loève Expansions of Lévy Processes

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Outline

We want to expand a continuous time stochastic process X in a stochastic Fourier series on the interval $[0,T]$:

$$X_t = \sum_{k\geq 1} Y_k \phi_k(t),$$

where $\{\phi_k\}_{k\geq 1}$ is an orthonormal basis of $L^2([0,T],\mathbb{R})$, and our stochastic Fourier coefficients are given by

$$Y_k := \int_0^T X_t \phi_k(t) \mathrm{d}t.$$

The main idea

Things to think about:

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- How should we choose $\{\phi_k\}_{k\geq 1}$?
- Distribution of, and dependence among $\{Y_k\}_{k\geq 1}$?

Assumptions: (a) $\mathbb{E}[X_t] = 0$, (b) $\mathbb{E}[X_t^2] < \infty$, and (c) $\mathrm{Cov}(X_s, X_t)$ is continuous

Basis: The eigenfunctions $\{e_k\}_{k\geq 1}$ corresponding to the non-zero eigenvalues $\{\lambda_k\}_{k\geq 1}$ of the operator $K : L^2([0,T]) \to L^2([0,T])$,

$$(Kf)(s) := \int_0^T \mathrm{Cov}(X_s, X_t) f(t) \mathrm{d}t$$

constitute a basis for $L^2([0,T])$. We define

$$Z_k := \int_0^T X_t e_k(t) \mathrm{d}t$$

and determine the order of $\{e_k\}_{k\geq 1}$, $\{Z_k\}_{k\geq 1}$, and $\{\lambda_k\}_{k\geq 1}$ according to $\lambda_1 \geq \lambda_2 \geq \lambda_3 \geq \ldots$.

Theorem (The Karhunen-Loève Theorem (KLT))

(i)

$$\mathbb{E}\left[\left(X_t - \sum_{k=1}^{d} Z_k e_k(t)\right)^2\right] \to 0, \quad \text{as} \quad d \to \infty$$

uniformly for $t \in [0,T]$. Additionally, the $\{Z_k\}_{k\geq 1}$ are uncorrelated and satisfy $\mathbb{E}[Z_k] = 0$ and $\mathbb{E}[Z_k^2] = \lambda_k$.

(ii) For any other basis $\{\phi_k\}_{k\geq 1}$ with corresponding Fourier coefficients $\{Y_k\}_{k\geq 1}$, and any $d \in \mathbb{N}$, we have

$$\int_0^T \mathbb{E}\left[(\varepsilon_d(t))^2\right] \mathrm{d}t \leq \int_0^T \mathbb{E}\left[(\tilde{\varepsilon}_d(t))^2\right] \mathrm{d}t,$$

where ε_d and $\tilde{\varepsilon}_d$ are the remainders $\varepsilon_d(t) := \sum_{d+1}^{\infty} Z_k e_k(t)$ and $\tilde{\varepsilon}_d(t) := \sum_{d+1}^{\infty} Y_k \phi_k(t)$.

Proof in:

GHANEM, R. G.. AND SPANOS, P.D.. (1991).
Stochastic finite elements: A spectral approach.
Springer–Verlag, New York–Berlin–Heidelberg.

They credit:

KAC, M. AND SIEGERT, A. (1947).
An explicit representation of a stationary Gaussian process.
Ann. Math. Stat. **18,** 438–442.

KARHUNEN, K. (1947).
Über lineare Methoden in der Wahrscheinlichkeitsrechnung.
Amer. Acad. Sc. Fennicade, Ser. A, I **37,** 3–79.

LOÉVE, M. (1948).
Fonctions aleatoires du second ordre.
In *Processus stochastic et mouvement Brownien*. ed. P. Lévy.
Gauthier Villars, Paris.

In order to use the KLT for a chosen process X in any meaningful way we see that we need to determine:

- The eigenvalues $\{\lambda_k\}_{k\geq 1}$ and eigenfunctions $\{e_k\}_{k\geq 1}$ of the operator K. I.e. solve:

$$\int_0^T \mathsf{Cov}(X_s, X_t) e_k(s) \mathsf{d}s = \lambda_k e_k(t),$$

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- The distribution of the Fourier coefficients $\{Z_k\}_{k\geq 1}$
- If we want to simulate an approximate path of X via a Karhunen Loève expansion (KLE) we also need to know how to simulate the first d Fourier coefficients, i.e. simulate the random vector

$$Z^{(d)} = (Z_1, \ldots, Z_d).$$

  Note: The components of this vector need not be independent!

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  Phoon, K.K.., Huang, H.W.. and Quek, S.T.. (2005).
  Simulation of strongly non-Gaussian processes using Karhunen–Loeve expansion.
  Probabilistic Engineering Mechanics **20,** 188–198.

Goal: For a one-dimensional Lévy process X satisfying $\mathbb{E}[X_t] = 0$ and $\mathbb{E}[X_t^2] < \infty$ determine

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- The distribution and dependence structure of $Z^{(d)} = (Z_1, \ldots, Z_d)$
- How to simluate $Z^{(d)}$

- The distribution of a d-dimensional Lévy process X or infinitely divisible vector ξ is completely determined by the *characteristic exponent*:

$$\Psi_X(\mathbf{z}) := -\frac{1}{t}\log\mathbb{E}[e^{\mathrm{i}\langle\mathbf{z},X_t\rangle}], \qquad \text{or} \quad \Psi_\xi(\mathbf{z}) := -\log\mathbb{E}[e^{\mathrm{i}\langle\mathbf{z},\xi\rangle}].$$

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- The characteristic exponent always has the form

$$\Psi(\mathbf{z}) = \frac{1}{2}\mathbf{z}^{\mathsf{T}}Q\mathbf{z} - \mathsf{i}\langle\mathbf{a},\mathbf{z}\rangle - \int_{\mathbb{R}^n\setminus\{\mathbf{0}\}} e^{\mathsf{i}\langle\mathbf{z},\mathbf{x}\rangle} - 1 - \mathsf{i}\langle\mathbf{z},\mathbf{x}\rangle h(\mathbf{x})\nu(\mathrm{d}\mathbf{x}).$$

So Ψ and therefore the distribution are determined by the generating triple $(\mathbf{a}, Q, \nu)_h$ where

- $\mathbf{a} \in \mathbb{R}^d$ $(a \in \mathbb{R})$,
- $Q \in \mathbb{R}^{d \times d}$ is positive semi definite $(\sigma^2 \geq 0)$ (*Gaussian component*), and
- ν is a measure (*Lévy measure*)
- h a *cut-off* function needed in general to make the integral converge

If

$$\int_{|\mathbf{x}|<1} |\mathbf{x}| \nu(\mathrm{d}\mathbf{x}) < \infty$$

then we can set $h \equiv 0$. (X has bounded variation when $Q = 0$ or $\sigma^2 = 0$.)

A scaled Brownian motion with drift with $\mathbb{E}[X_t] = \mu t$ and $\mathrm{Var}(X_t) = \sigma^2 t$ has characteristic exponent

$$\Psi(z) = \frac{\sigma^2}{2} z^2 - i\mu z.$$

Since $\nu \equiv 0$ there are no jumps and we have generating triple $(\mu, \sigma^2, 0)$

For $c,\ \rho,\ \hat{\rho} > 0$ set

$$\nu(\mathsf{d}x) = \mathbb{I}(x<0)c\frac{e^{\hat{\rho}x}}{|x|}\mathsf{d}x + \mathbb{I}(x<0)c\frac{e^{-\rho x}}{x}\mathsf{d}x.$$

Then

$$\Psi(z) = -\int_{\mathbb{R}\setminus\{0\}} \left(e^{\mathsf{i}zx} - 1\right)\nu(\mathsf{d}x) = c\left(\log\left(1 - \frac{\mathsf{i}z}{\rho}\right) + \log\left(1 + \frac{\mathsf{i}z}{\hat{\rho}}\right)\right)$$

is the characteristic exponent of a variance gamma process with generating triple $(0, 0, \nu)$.

Under the KLT assumption it is straight-forward to show that

$$\mathsf{Var}(X_t) = \mathbb{E}[X_t^2] = \alpha t, \quad \text{and} \quad \mathsf{Cov}(X_s, X_t) = \alpha \min(s,t),$$

where $\alpha = \Psi''(0)$.

For example, for our Brownian motion we have $\alpha = \sigma^2$ and for our variance gamma process $\alpha = c(\rho^{-2} + \hat{\rho}^{-2})$.

But this shows that the eigenvalues/functions of the operator K for a Lévy process can be derived in exactly the same manner as those of a Brownian motion. And so we get our first result essentially for free...

Proposition

The eigenvalues and associated eigenfunctions of the operator K defined with respect to a Lévy process X are given by

$$\lambda_k = \frac{\alpha T^2}{\pi^2 \left(k - \frac{1}{2}\right)^2}, \quad \text{and} \quad e_k(t) = \sqrt{\frac{2}{T}} \sin\left(\frac{\pi}{T}\left(k - \frac{1}{2}\right) t\right),$$

for $k \in \mathbb{N}$ and $t \in [0, T]$.

Idea of the Proof: Differentiate both sides of

$$\alpha \int_0^T \min(s,t) e_k(s) \mathrm{d}s = \lambda_k e_k(t),$$

twice with respect to t to reduce to an ODE. Details for Brownian Motion case in

Ash, R.B.. and Gardner, M.F.. (1975).
Topics in stochastic process.
Academic Press, New York–San Francisco–London.

Define the *total variance* of X on $[0,T]$ as

$$v(T) := \int_0^T \mathbb{E}[X_t^2]\mathrm{d}t = \frac{\alpha T^2}{2}$$

But we also have $v(T) = \sum_{k\geq 1} \lambda_k$ (in general). Therefore, the total variance explained by a d-term approximation is

$$\frac{\sum_{k=1}^d \lambda_k}{v(T)} = \frac{2}{\pi^2} \sum_{k=1}^{d} \frac{1}{\left(k - \frac{1}{2}\right)^2}.$$

Computation yields: the first 2, 5 and 21 terms already explain 90%, 95%, and 99% of the total variance of the process. Holds independently of X, α or T.

A key property of Gaussian processes, is that the coefficients $\{Z_k\}_{k\geq 1}$ are again Gaussian. For example,

$$B_t = \sqrt{2}\sum_{k\geq 1} Z_k \frac{\sin\left(\pi(k-\frac{1}{2}t)\right)}{\pi\left(k-\frac{1}{2}\right)}$$

where B is a standard Brownian Motion on $[0,1]$ and the $\{Z_k\}_{k\geq 1}$ are i.i.d. random variables with common distribution $\mathcal{N}(0,1)$.

Lemma

Let X be a Lévy process and $\{f_k\}_{k=1}^d$ be a collection of functions which are in $L^1([0,T])$. Then the vector ξ consisting of elements

$$\xi_k = \int_0^T X_t f_k(s)\mathrm{d}s, \quad k \in \{1,2,\ldots,d\},$$

has an ID distribution with characteristic exponent

$$\Psi_\xi(\mathbf{z}) = \int_0^T \Psi_X\left(\langle \mathbf{z}, \mathbf{u}(t)\rangle\right)\mathrm{d}t, \quad \mathbf{z} \in \mathbb{R}^d,$$

where $\mathbf{u} : [0,T] \to \mathbb{R}^d$ is the function with k-th component $u_k(t) := \int_t^T f_k(s)\mathrm{d}s$, $k \in \{1,2,\ldots,d\}$.

Theorem

If X is a Lévy process with generating triple (a, σ^2, ν) that satisfies the KLT and BV assumptions then $Z^{(d)}$ has generating triple $(\mathbf{a}, \mathcal{Q}, \Pi)$ where $\mathbf{a}$ is the vector with entries

$$a_k := a\frac{(-1)^{k+1}\sqrt{2}T^{\frac{3}{2}}}{\pi^2\left(k-\frac{1}{2}\right)^2}, \quad k \in \{1, 2, \ldots, d\},$$

Theorem (cont.)

$\mathcal{Q}$ is a diagonal $d \times d$ matrix with entries

$$q_{k,k} := \frac{\sigma^2}{2} \frac{T^2}{\pi^2 \left(k - \frac{1}{2}\right)^2}, \quad k \in \{1, 2, \dots, d\},$$

and Π is the measure,

$$\Pi(B) := \int_{\mathbb{R}\setminus\{0\}\times[0,T]} \mathbb{I}(f(\mathbf{v}) \in B)(\nu \times \lambda)(\mathbf{dv}), \quad B \in \mathcal{B}_{\mathbb{R}^d \setminus \{\mathbf{0}\}},$$

where λ is the Lebesgue measure on $[0, T]$ and $f : \mathbb{R} \times [0, T] \to \mathbb{R}^d$ is the function

$$(x, t) \mapsto \frac{\sqrt{2T}x}{\pi} \left(\frac{\cos\left(\frac{\pi}{T}\left(1 - \frac{1}{2}\right)t\right)}{\left(1 - \frac{1}{2}\right)}, \dots, \frac{\cos\left(\frac{\pi}{T}\left(d - \frac{1}{2}\right)t\right)}{\left(d - \frac{1}{2}\right)} \right)^{\mathsf{T}}.$$

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- From Lemma: $\Psi_{Z^{(d)}}(z) = \int_0^T \Psi_X(\langle \mathbf{z}, \mathbf{u}(t) \rangle) \mathrm{d}t$

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$$u_k(t) = \sqrt{\frac{2}{T}} \int_t^T \sin\left(\frac{\pi}{T}\left(k - \frac{1}{2}\right) s\right) \mathrm{d}s = \sqrt{2T} \frac{\cos\left(\frac{\pi}{T}\left(k-\frac{1}{2}\right) t\right)}{\pi(k - \frac{1}{2})}$$

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- Evaluation of integral over $[0, T]$ and the orthogonality of $\{u_k\}_{1 \leq k \leq d}$ give $\mathbf{a}$ and $\mathcal{Q}$.

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- Evaluation of integral over $[0, T]$ and the orthogonality of $\{u_k\}_{1\le k\le d}$ give $\mathbf{a}$ and $\mathcal{Q}$.
- Fubini's theorem and a change of variables yields Π.

Corollary

$Z^{(d)}$ has independent entries if, and only if, ν is the zero measure.

Idea of Proof:

- ($\Leftarrow$) known, or because $\mathcal{Q}$ is diagonal
- ($\Rightarrow$) we show that Π is not supported by the coordinate axes

How can we simulate a random vector with dependent entries with only knowledge of the characteristic function?

In general, this seems to be a difficult problem. Infinite divisbility makes it possible!

Idea: Write $Z^{(d)}$ as a(n) (infinite) sum of simpler random vectors.

- Random sequences $\{V_i\}_{i\geq 1}$ and $\{\Gamma_i\}_{i\geq 1}$ which are independent of each other and defined on a common probability space.

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- $\{V_i\}_{i\geq 1}$ are independent and take values in some measurable space D with common dist. F
- $H : (0, \infty) \times D \to \mathbb{R}^d$, measurable, and

$$S_n := \sum_{i=1}^{n} H(\Gamma_i, V_i), \quad n \in \mathbb{N},$$

and

$$\mu(B) := \int_0^\infty \int_D \mathbb{I}(H(r,v) \in B) F(\mathrm{d}v)\mathrm{d}r, \quad B \in \mathcal{B}_{\mathbb{R}^d \setminus \{\mathbf{0}\}}.$$

Theorem

If μ is a Lévy measure satifying the BV assumption, then S_n converges almost surely to an ID random vector with generating triple $(\mathbf{0}, \mathbf{0}, \mu)$ as $n \to \infty$.

Theorems 3.1, 3.2, and 3.4 in

ROSIŃSKI, J. (1990).
On series representations of infinitely divisible random vectors.
The Annals of Probability **18**, 405–430.

Idea: Show that Π has a disintegrated form like μ.

Simplifying assumptions:

- X has no Gaussian component
- X has only positive jumps
- $X_t \overset{d}{=} X_t^+ - X_t^- + B_t \Rightarrow Z_X^{(d)} \overset{d}{=} Z_{X^+}^{(d)} - Z_{X^-}^{(d)} + Z_B^{(d)}$

Necessary assumption: ν has a strictly positive density π.

We want

$$g(x) := \int_x^\infty \pi(s)\mathrm{d}s$$

to have a well defined, non-increasing inverse g^{-1}.

Theorem

If X is a Lévy process with only positive jumps, satisfying the KLT and BV assumptions, such that X has triple $(a, 0, \pi)$, where π is strictly positive, then

$$Z^{(d)} \stackrel{d}{=} \mathbf{a} + \sum_{i \geq 1} H(\Gamma_i, U_i),$$

where $\{U_i\}_{i \geq 1}$ is an i.i.d. sequence of uniform random variables on $[0, 1]$, and

$$H(r, v) := f(g^{-1}(r/T), Tv),$$

where f is the function defined previously, i.e.

$$f(x, t) = \frac{\sqrt{2T}x}{\pi} \left(\frac{\cos\left(\frac{\pi}{T}\left(1 - \frac{1}{2}\right)t\right)}{\left(1 - \frac{1}{2}\right)}, \ldots, \frac{\cos\left(\frac{\pi}{T}\left(d - \frac{1}{2}\right)t\right)}{\left(d - \frac{1}{2}\right)} \right)^{\mathsf{T}}.$$

$$S_t^{(d)} := \sqrt{\frac{2}{T}} \sum_{k=1}^{d} Z_k \sin\left(\frac{\pi}{T}\left(k - \frac{1}{2}\right) t\right)$$

$$Z_k \stackrel{d}{=} a_k + \sum_{i \geq 1} \frac{\sqrt{2T} g^{-1}(\Gamma_i/T)}{\pi} \frac{\cos\left(\pi\left(k - \frac{1}{2}\right) U_i\right)}{\left(k - \frac{1}{2}\right)},$$

Nice/Unique features:

- $Z^{(d)}$ is independent of t

Potentially difficult:

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Discussion

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- Inverting g

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- Inverting g
- Convergence of sum depends on decay of g^{-1}

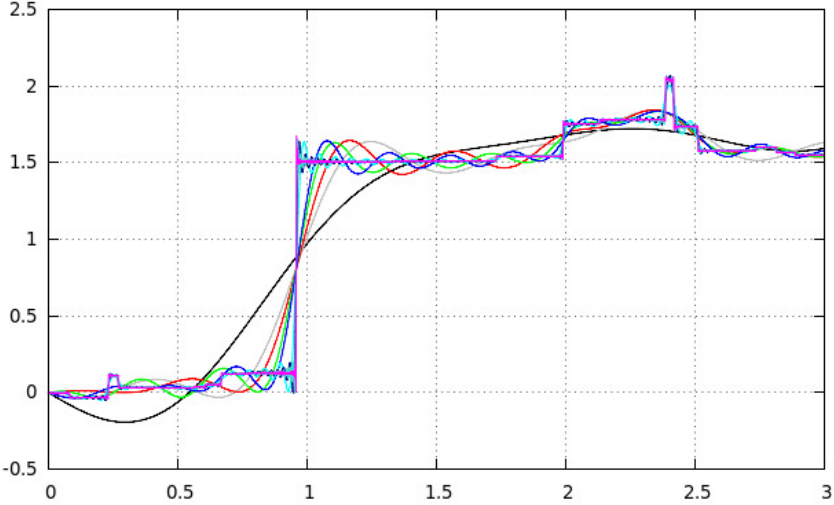
Consider a VG process with $c = 1$, $\rho = 2$, $\hat{\rho} = 5$, and $T = 3$. The process X^+ has Lévy measure $\nu(\mathrm{d}x) = \frac{e^{-\rho x}}{x}\mathrm{d}x$ so that we have

$$g(x) = c\int_x^\infty \frac{e^{-\rho x}}{x}\mathrm{d}x = cE_1(\rho x), \quad \text{and} \quad g^{-1}\left(\frac{\Gamma_i}{T}\right) = \frac{1}{\rho}E_1^{-1}\left(\frac{\Gamma_i}{Tc}\right),$$

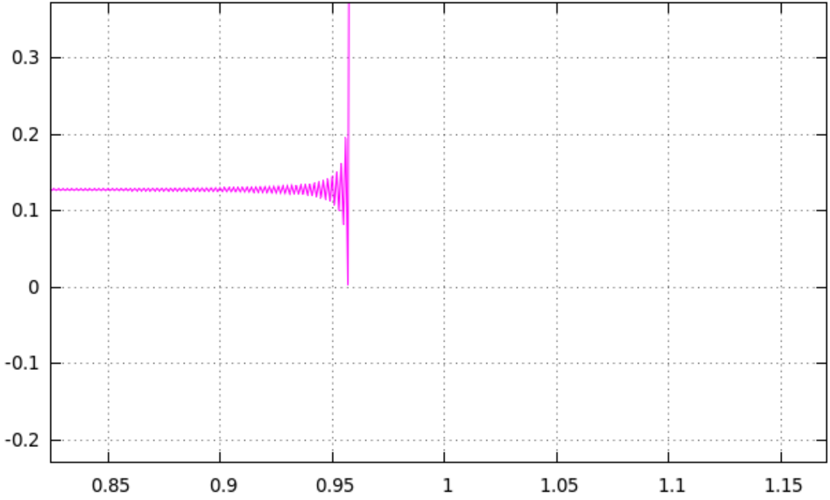
where E_1 is the exponential integral function.

We truncate the random series when $\Gamma_i/(Tc) > 46$ since at this point $g^{-1}(\Gamma_i/T) < \rho^{-1}10^{-19}$. We expect to generate $46Tc = 138$ random pairs (U_i, Γ_i) per path of X^+.

Sample paths of $S^{(d)}$ with $d \in \{5, 10, 15, 20, 25, 100, 250, 500, 3000\}$

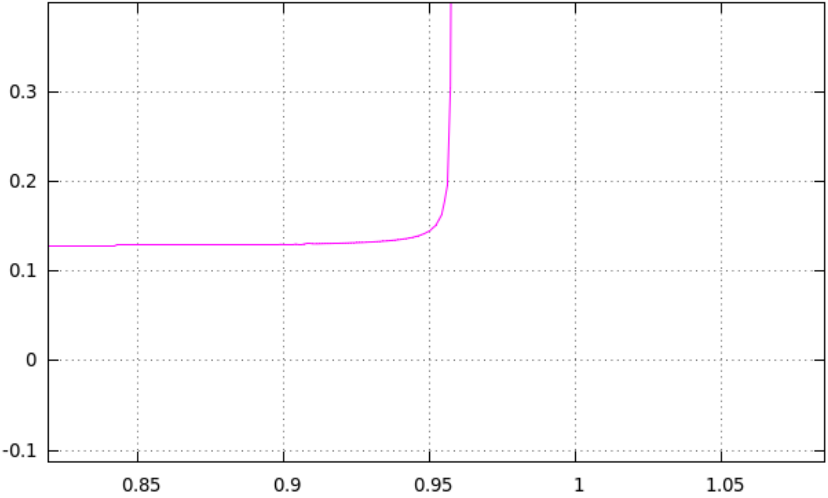


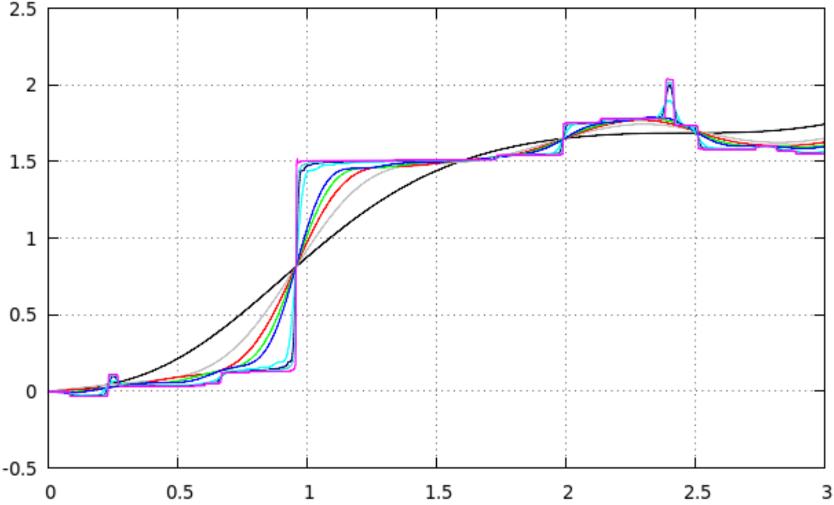
A closer look at $d = 3000$ shows...



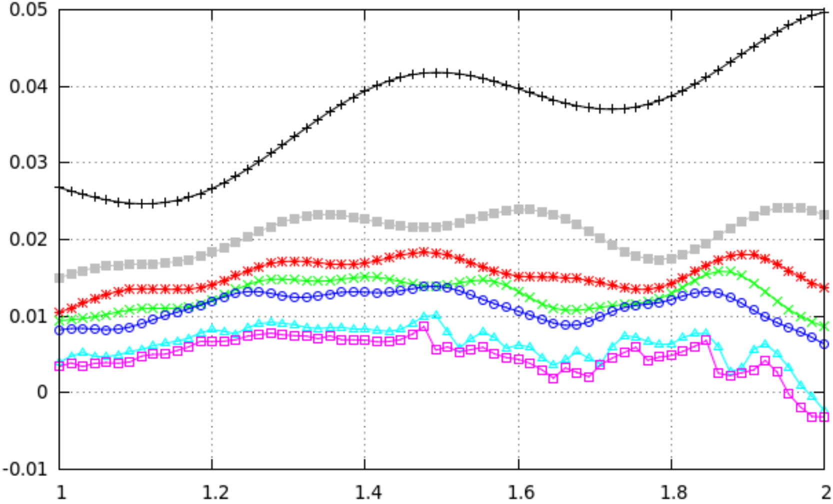
... the Gibbs phenomenon...

... but we can remove this if we want. For example if we form the Cesàro sums $C_t^{(d)} := \frac{1}{d}\sum_{k=1}^{d} S_t^{(k)}$ the Gibbs phenomenon disappears.




2.5
2
1.5
1
0.5
0
-0.5
0
0.5
1
1.5
2
2.5
3

Suppose we want to compute $\mathbb{E}[e^{X_t}] = e^{-\Psi(-\mathrm{i})} = (5/3)^t$ on the interval $[1, 2]$ by simulating 10^6 paths of $S^{(d)}$. The errors:



This method could be useful if we have to evaluate $\mathbb{E}[f(X_t)]$ at many different points in the interval $[0, T]$, or at previously unknown points, say if we wanted to find the minimum in t.

If, for example, $d \leq 100$ is deemed good enough, then 100×10^6 single precision floating point numbers (realizations of the $\{Z_k\}_{1 \leq k \leq d}$) can be stored in memory: 4 Bytes $\times 10^8 \approx 0.37$ GB. Even $d = 3000$ is not impossible: 4 Bytes $\times 3 \times 10^9 \approx 11.2$ GB.

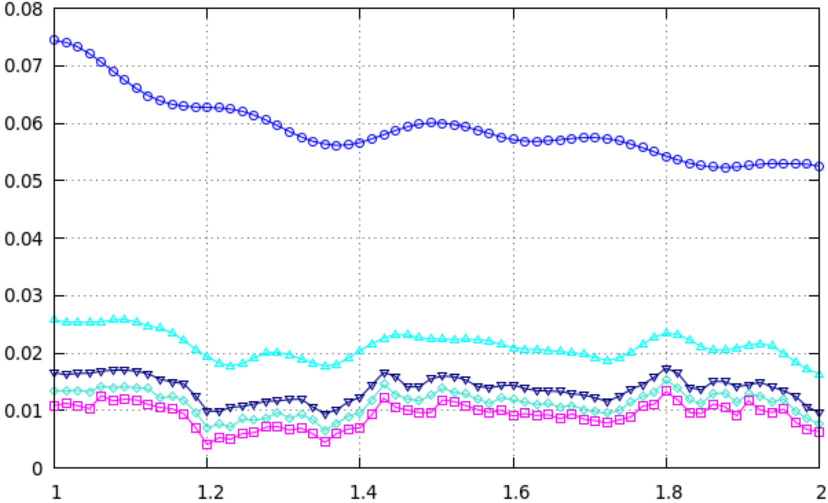
If we want to compute $\mathbb{E}[f(X_t)]$ at points $t_1 \leq t_2 \leq \ldots \leq t_d$ in $[0, T]$ then we can also use a fractional FFT algorithm to compute the realizations of $\{S_{t_k}^{(d)}\}_{1 \leq k \leq d}$ with $O(d \log(d))$ operations.

Consider a VG process X with $c = 5$, $\rho = 21.8735$, $\hat{\rho} = 56.4414$, and $T = 3$. We will also add a drift μ whose value is to be determined. Suppose we want to compute

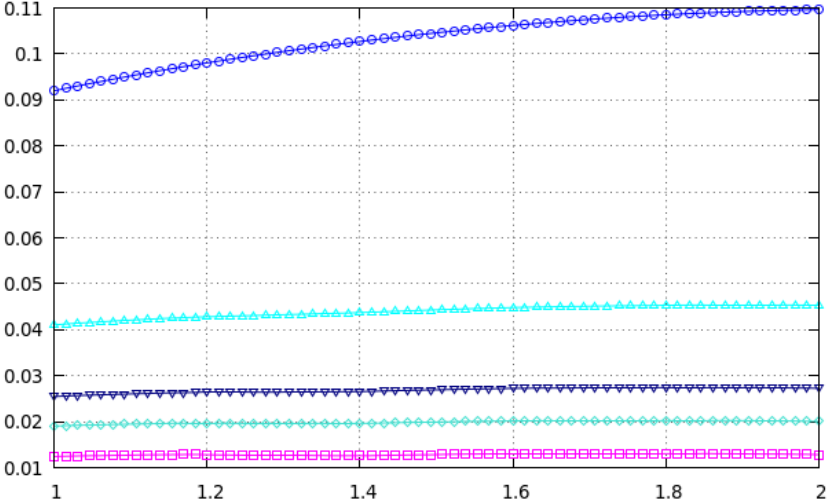
$$e^{-r\tau}\mathbb{E}\left[\left(A_0 \exp\left(X_\tau\right) - K\right)^+\right], \quad \text{or}$$
$$e^{-r\tau}\mathbb{E}\left[\left(\frac{A_0}{\tau}\int_0^\tau \exp\left(X_t\right) \mathrm{d}t - K\right)^+\right],$$

where r, $A_0, K > 0$. Then μ is determined by the risk neutral condition $\Psi(-\mathrm{i}) = -r$, and the expressions represent the price of a European and Asian call option respectively.

Replace X by $S^{(d)}$ and do Monte Carlo with 10^6 iterations. Errors for $\tau \in [1, 2]$ compared to a benchmark computed using a Fourier transform technique.



Replace X by $C^{(d)}$ and do Monte Carlo with 10^6 iterations. Errors for $\tau \in [1, 2]$ compared to a benchmark computed using a Fourier transform technique.



Let $c^{(d)}$ be a realization of $C^{(d)}$ on $[0, T]$. Then $t \mapsto \exp(c_t^{(d)})$ is smooth function of t. Applying the right change of variables $\cos(\theta) = 2\left(\frac{t}{\tau} - 1\right)$ transforms

$$\int_0^\tau \exp\left(c_t^{(d)}\right) \mathrm{d}t$$

into the integral of a periodic function over $[0, \pi]$. We expect exponential convergence of the trapezoidal rule for such a function. Numerical evaluation is then possible via the Clenshaw-Curtis quadrature.

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