Approximating Lévy processes with completely monotone jumps

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Joint work with Alexey Kuznetsov.
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Definitions and notations

A Lévy process $X$ is specified by the triple $(a, \sigma^2, \Pi)$, where $a \in \mathbb{R}$, $\sigma \geq 0$ and $\Pi(dx)$ satisfies $\int_{\mathbb{R}} \min(1, x^2) \Pi(dx) < \infty$.

The Laplace exponent $\psi(z)$ is defined as

$$\mathbb{E} \left[ e^{zX_t} \right] = e^{t\psi(z)}, \quad \text{Re}(z) = 0.$$ 

The Lévy-Khintchine representation for $\psi(z)$ is

$$\psi(z) = \frac{\sigma^2 z^2}{2} + az + \int_{\mathbb{R}} \left( e^{zx} - 1 - zx \mathbf{1}_{\{|x|<1\}} \right) \Pi(dx).$$
Definitions and notations

- We define the supremum $\overline{X}_t = \sup\{X_s : 0 \leq s \leq t\}$ and similarly for the infimum $\underline{X}_t$;
- $e(q)$ denotes an exponential random variable (with mean $1/q$), independent of $X$;
- Define $S_q = \overline{X}e(q)$ and $I_q = \underline{X}e(q)$
- The Wiener-Hopf factors are defined as $\phi^+_q(z) = \mathbb{E}[\exp(-zS_q)]$ and $\phi^-_q(z) = \mathbb{E}[\exp(zI_q)]$;
Wiener-Hopf factorization

- $X_{e(q)} - S_q$ is independent of $S_q$ and has the same distribution as $I_q$. 
Wiener-Hopf factorization

- \( X_{e(q)} - S_q \) is independent of \( S_q \) and has the same distribution as \( I_q \).

\[
\frac{q}{q - \psi(z)} = \phi_q^+(-z)\phi_q^-(z),
\]

since

\[
\frac{q}{q - \psi(z)} = \mathbb{E} \left[ e^{zX_{e(q)}} \right] = \mathbb{E} \left[ e^{z(X_{e(q)} - S_q) + zS_q} \right]
\]
Wiener-Hopf factorization

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  \]

- If we can factorize $q/(q - \psi(z))$ as a product of two functions $f^\pm(z)$, such that $f^+(z)$ \{resp. $f^-(z)$\} is analytic and zero-free in the half-plane $\text{Re}(z) > 0$ \{resp. $\text{Re}(z) < 0$\}, (plus some growth conditions) - then we can identify $\phi_q^\pm(z) = f^\pm(z)$. 
Applications: Math finance

- We want processes with jumps of infinite activity (and sometimes of infinite variation).
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- In order to price European options we need to have explicit formulas for the Laplace exponent $\psi(z)$. 


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  Jeannin, M. and Pistorius, M.,
  A transform approach to calculate prices and greeks of barrier options driven by a class of Lévy processes.
### Popular processes in mathematical finance

<table>
<thead>
<tr>
<th></th>
<th>Variance Gamma (VG)</th>
<th>Normal Inverse Gaussian (NIG)</th>
<th>Generalized Tempered Stable (CGMY or KoBol)</th>
<th>Hyper-exponential</th>
</tr>
</thead>
<tbody>
<tr>
<td><strong>Activity</strong></td>
<td>Infinite</td>
<td>Infinite</td>
<td>Parameter dependent</td>
<td>Finite</td>
</tr>
<tr>
<td><strong>Variation</strong></td>
<td>Finite</td>
<td>Infinite</td>
<td>Parameter dependent</td>
<td>Finite</td>
</tr>
<tr>
<td><strong>WHF</strong></td>
<td>No explicit form</td>
<td>No explicit form</td>
<td>No explicit form</td>
<td>Rational function</td>
</tr>
</tbody>
</table>

E.g.: The Laplace exponent of the VG process:

\[
ψ(z) = zγ + \frac{1}{k} \ln \left( 1 - \frac{σ^2 k}{2} z^2 - \theta k z \right).
\]
Completely monotone jumps

Definition

A function $f(x)$ is called completely monotone if $(-1)^n f^{(n)}(x) > 0$ for all $x > 0$, $n = 0, 1, 2, \ldots$.

Definition

A Lévy process has completely monotone jumps, if $\Pi(dx)$ is absolutely continuous with density $\pi(x)$, and $\pi(x)$ and $\pi(-x)$ are completely monotone for $x \in (0, \infty)$.

Theorem

The jump density of a process $X$ is completely monotone if and only if $S_q$ and $I_q$ are mixtures of exponentials.

L.C.G. Rogers.

Wiener-Hopf factorization of diffusions and Lévy processes.

Hyperexponential processes

The density of the Lévy measure is

\[ \pi(x) = \mathbb{I}_{\{x>0\}} \sum_{i=1}^{N} a_i \rho_i e^{-\rho_i x} + \mathbb{I}_{\{x<0\}} \sum_{i=1}^{\hat{N}} \hat{a}_i \hat{\rho}_i e^{\hat{\rho}_i x}, \]

where all the coefficients are positive.

The Laplace exponent is a rational function

\[ \psi(z) = \frac{\sigma^2}{2} z^2 + \mu z + z \sum_{i=1}^{N} \frac{a_i}{\rho_i - z} - z \sum_{i=1}^{\hat{N}} \frac{\hat{a}_i}{\hat{\rho}_i + z}. \]
Hyperexponential processes

Assume $\sigma > 0$.

- The Wiener-Hopf factors are given by

$$
\phi^+_q(z) = \frac{1}{1 + \frac{z}{\zeta_1}} \prod_{i=1}^{N} \frac{1 + \frac{z}{\rho_i}}{1 + \frac{z}{\zeta_{i+1}}}, \quad \phi^-_q(z) = \frac{1}{1 + \frac{z}{\hat{\zeta}_1}} \prod_{i=1}^{\hat{N}} \frac{1 + \frac{z}{\hat{\rho}_i}}{1 + \frac{z}{\hat{\zeta}_{i+1}}},
$$

where $\zeta_i$ and $\hat{\zeta}_i$ are the (real) solutions to $\psi(z) = q$.

- The distribution of $S_q$ is a mixture of exponentials

$$
\frac{d}{dx} \mathbb{P}(S_q \leq x) = \sum_{i=1}^{N+1} c_i \zeta_i e^{-\zeta_i x},
$$

where $c_i > 0$ and $\sum c_i = 1$, and similarly for $I_q$. 
Processes with hyper-exponential jumps are great to work with, but...
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we have a problem: we can’t have jumps of infinite activity/infinite variation.
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The other processes are completely monotone and have infinite activity, but we do not have closed form expressions for the Wiener-Hopf factors.
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The other processes are completely monotone and have infinite activity, but we do not have closed form expressions for the Wiener-Hopf factors.

How do we approximate a general Lévy process with completely monotone jumps by a hyperexponential process?
Outline

1 Introduction

2 Theoretical results

3 Numerical results
Main idea

- Approximating a Lévy process is equivalent to approximating its Laplace exponent $\psi(z)$. 
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- The Laplace exponent of a hyperexponential process is a rational function.
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- Thus we have two problems:
  1. Approximate $\psi(z)$ by a rational function $\tilde{\psi}(z)$,
Main idea

- Approximating a Lévy process is equivalent to approximating its Laplace exponent $\psi(z)$.
- The Laplace exponent of a hyperexponential process is a rational function.
- Thus we have two problems:
  1. Approximate $\psi(z)$ by a rational function $\tilde{\psi}(z)$,
  2. Show that $\tilde{\psi}(z)$ is itself a Laplace exponent of a Lévy process.
Padé approximation

**Definition**

Let $f$ be a function with a power series representation $f(z) = \sum_{i=0}^{\infty} c_i z^i$. If there exist polynomials $P_m(z)$ and $Q_n(z)$ satisfying $\deg(P) \leq m$, $\deg(Q) \leq n$, $Q_n(0) = 1$ and

$$\frac{P_m(z)}{Q_n(z)} = c_0 + c_1 z + \cdots + c_{m+n} z^{m+n} + O(z^{m+n+1}), \quad z \to 0,$$

then we say that $f\left[\frac{m}{n}\right](z) := \frac{P_m(z)}{Q_n(z)}$ is the $[m/n]$ Padé approximant of $f$. 

Daniel Hackmann

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# A simple example of Padé approximations

<table>
<thead>
<tr>
<th>$m$ \ $n$</th>
<th>0</th>
<th>1</th>
<th>2</th>
<th>3</th>
</tr>
</thead>
<tbody>
<tr>
<td>0</td>
<td>$\frac{1}{1}$</td>
<td>$\frac{1}{1-z}$</td>
<td>$\frac{1}{1-z + \frac{1}{2}z^2}$</td>
<td>$\frac{1}{1-z + \frac{1}{2}z^2 - \frac{1}{6}z^3}$</td>
</tr>
<tr>
<td>1</td>
<td>$\frac{1+z}{1}$</td>
<td>$\frac{1 + \frac{1}{2}z}{1 - \frac{1}{2}z}$</td>
<td>$\frac{1 + \frac{1}{3}z + \frac{1}{6}z^2}{1 - \frac{1}{3}z + \frac{1}{6}z^2}$</td>
<td>$\frac{1 + \frac{1}{4}z}{1 - \frac{3}{4}z + \frac{1}{4}z^2 - \frac{1}{12}z^3}$</td>
</tr>
<tr>
<td>2</td>
<td>$\frac{1 + \frac{1}{2}z^2}{1}$</td>
<td>$\frac{1 + \frac{3}{4}z + \frac{1}{4}z^2 + \frac{1}{24}z^3}{1 - \frac{3}{4}z + \frac{1}{24}z^2}$</td>
<td>$\frac{1 + \frac{3}{5}z + \frac{3}{20}z^2 + \frac{1}{60}z^3}{1 - \frac{3}{5}z + \frac{3}{20}z^2 - \frac{1}{120}z^3}$</td>
<td>$\frac{1 + \frac{1}{2}z + \frac{1}{10}z^2 + \frac{1}{120}z^3}{1 - \frac{1}{2}z + \frac{1}{10}z^2 - \frac{1}{120}z^3}$</td>
</tr>
<tr>
<td>3</td>
<td>$\frac{1 + \frac{1}{3}z^2 + \frac{1}{6}z^3}{1}$</td>
<td>$\frac{1 + \frac{3}{4}z + \frac{1}{4}z^2 + \frac{1}{24}z^3}{1 - \frac{3}{4}z + \frac{1}{24}z^2}$</td>
<td>$\frac{1 + \frac{3}{5}z + \frac{3}{20}z^2 + \frac{1}{60}z^3}{1 - \frac{3}{5}z + \frac{3}{20}z^2 - \frac{1}{120}z^3}$</td>
<td>$\frac{1 + \frac{1}{2}z + \frac{1}{10}z^2 + \frac{1}{120}z^3}{1 - \frac{1}{2}z + \frac{1}{10}z^2 - \frac{1}{120}z^3}$</td>
</tr>
<tr>
<td>4</td>
<td>$\frac{1 + \frac{1}{4}z^2 + \frac{1}{6}z^3 + \frac{1}{24}z^4}{1}$</td>
<td>$\frac{1 + \frac{3}{5}z + \frac{3}{10}z^2 + \frac{1}{15}z^3 + \frac{1}{720}z^4}{1 - \frac{3}{5}z + \frac{3}{10}z^2 - \frac{1}{720}z^3}$</td>
<td>$\frac{1 + \frac{1}{2}z + \frac{1}{10}z^2 + \frac{1}{120}z^3}{1 - \frac{1}{2}z + \frac{1}{10}z^2 - \frac{1}{120}z^3}$</td>
<td>$\frac{1 + \frac{1}{4}z^2 + \frac{1}{10}z^3 + \frac{1}{120}z^3}{1 - \frac{1}{4}z^2 - \frac{1}{120}z^3}$</td>
</tr>
</tbody>
</table>

**Figure:** The initial part of the Padé table for $e^z$
Gaussian quadrature

- $\nu$ is a finite positive measure on a closed bounded interval $[a, b]$
Gaussian quadrature

- $\nu$ is a finite positive measure on a closed bounded interval $[a, b]$
- For each $n$ we want to find a measure $\tilde{\nu}_n$ on a finite number of points in $[a, b]$ such that we match the first $2n - 1$ moments of $\nu$, i.e.

\[
\int_{[a,b]} x^j \nu(dx) = \sum_{i} x_i^j w_i, \quad \text{for } j = 1, \ldots, 2n - 1.
\]
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$$\int_{[a,b]} x^j \nu(dx) = \sum_{i}^{n} x_i^j w_i, \text{ for } j = 1, \ldots, 2n - 1.$$

- The points $x_i$ and $w_i$ are the nodes and weights of the Gaussian quadrature.
Gaussian quadrature and orthogonal polynomials

\[ \{p_n(x)\}_{n \geq 0} \text{ be the sequence of orthogonal polynomials with respect to the measure } \nu(dx): \deg(p_n) = n \text{ and} \]

\[ (p_n, p_m)_\nu := \int_{[a,b]} p_n(x)p_m(x)\nu(dx) = d_n\delta_{n,m} \]
Gaussian quadrature and orthogonal polynomials

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- The nodes of the Gaussian quadrature \(\tilde{\nu}_n\) are the zeros of \(p_n\) and the weights may be calculated from \(p_{n-1}, p_n\).
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G. Szegö.

*Orthogonal Polynomials.*

Main theorem (two-sided case)

**Assumption:** The Lévy measure $\Pi(dx)$ is absolutely continuous, and its density $\pi(x)$ is completely monotone and decreases exponentially fast as $x \to \pm \infty$.

Using Bernstein’s theorem, we see that there exists a positive measure $\mu$, with support in $\mathbb{R}\backslash\{0\}$, such that for all $x \in \mathbb{R}$

$$
\pi(x) = \mathbb{I}\{x>0\} \int_{(0,\infty)} e^{-ux} \mu(du) + \mathbb{I}\{x<0\} \int_{(-\infty,0)} e^{-ux} \mu(du).
$$

(1)

We denote

$$
\mu^*(A) = \mu(\{v \in \mathbb{R} : v^{-1} \in A\}).
$$

Then $|v|^3 \mu^*(dv)$ is a finite measure, with bounded support.
Main theorem (two-sided case)

Assume that $\sigma = 0$. Let $\{x_i\}_{1\leq i \leq n}$ and $\{w_i\}_{1\leq i \leq n}$ be the nodes and the weights of the Gaussian quadrature of order $n$ with respect to the measure $|v|^3 \mu^*(dv)$. We define

$$
\psi_n(z) := az + z^2 \sum_{i=1}^{n} \frac{w_i}{1 - zx_i}.
$$

**Theorem**

(i) The function $\psi_n(z)$ is the $[n + 1/n]$ Padé approximant of $\psi(z)$.

(ii) The function $\psi_n(z)$ is the Laplace exponent of a hyperexponential process $X^{(n)}$ having the characteristic triple $(a, \sigma_n^2, \pi_n)_{n\equiv x}$, where...
Main theorem (two-sided case)

Theorem

(ii)

\[
\pi_n(x) := \begin{cases} 
\sum_{1 \leq i \leq n : x_i < 0} w_i |x_i|^{-3} e^{-\frac{x}{x_i}}, & \text{if } x < 0, \\
\sum_{1 \leq i \leq n : x_i > 0} w_i x_i^{-3} e^{-\frac{x}{x_i}}, & \text{if } x > 0.
\end{cases}
\]

(iii) The random variables \(X_1^{(n)}\) and \(X_1\) satisfy \(\mathbb{E}[(X_1^{(n)})^j] = \mathbb{E}[(X_1)^j]\) for \(1 \leq j \leq 2n + 1\).
Convergence

Theorem

For any compact set $A \subset \mathbb{C} \setminus \{(-\infty, -\hat{\rho}] \cup [\rho, \infty)\}$ there exist $c_1 = c_1(A) > 0$ and $c_2 = c_2(A) > 0$ such that for all $z \in A$ and all $n \geq 1$

$$|\psi_n(z) - \psi(z)| < c_1 e^{-c_2 n}.$$
One-sided processes

- For CM subordinators, all three functions $\psi^{[n/n]}(z)$, $\psi^{[n+1/n]}(z)$, $\psi^{[n+2/n]}(z)$ are Laplace exponents of hyperexponential processes.
One-sided processes

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  $\psi[n+2/n](z)$ are Laplace exponents of hyperexponential processes.

- For CM spectrally-positive processes of infinite variation, only
two functions $\psi[n+1/n](z)$, $\psi[n+2/n](z)$ are Laplace exponents of
hyperexponential processes.
One-sided processes

- For CM subordinators, all three functions $\psi^{[n/n]}(z)$, $\psi^{[n+1/n]}(z)$, $\psi^{[n+2/n]}(z)$ are Laplace exponents of hyperexponential processes.

- For CM spectrally-positive processes of infinite variation, only two functions $\psi^{[n+1/n]}(z)$, $\psi^{[n+2/n]}(z)$ are Laplace exponents of hyperexponential processes.

- There exist explicit formulas for a number of important examples: In the VG case we have $\psi^{[n/n]}(z) = P_n(z)/Q_n(z)$, where

$$P_n(z) = 2 \sum_{j=0}^{n} \binom{n}{j}^2 [H_{n-j} - H_j] (1 - z)^j, \quad Q_n(z) = z^n P_n \left( \frac{2}{z} - 1 \right).$$

and $H_j := 1 + 1/2 + \cdots + 1/j$. 
How do we prove all these results?

- One can show that only $\psi^{[n/n]}(z)$, $\psi^{[n+1/n]}(z)$ and $\psi^{[n+2/n]}(z)$ can possibly be Laplace exponents of a Lévy process.
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- The Lévy-Khintchine formula + Fubini’s theorem + change of variables give us

$$
\psi(z) = \sigma^2 \frac{z^2}{2} + az + z^2 \int_{[a,b]} |v|^3 \mu^*(dv) \frac{1}{1-vz},
$$

where $a < 0 < b$. 

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\psi(z) = \frac{\sigma^2}{2} z^2 + a z + z^2 \int_{[a,b]} \frac{|v|^3 \mu^*(dv)}{1 - vz},
$$

where $a < 0 < b$.

- $\psi(z)$ is closely related to Stieltjes functions:

$$
f(z) := \int_{[0,R^{-1}]} \frac{\nu(du)}{1 + zu}
$$
Some more theory on Stieltjes functions.

- $f^{[m/n]}(z)$ always exists provided $m \geq n - 1$. 
Some more theory on Stieltjes functions.

- $f^{[m/n]}(z)$ always exists provided $m \geq n - 1$.
- The poles of $f^{[m/n]}(z)$ are simple, real and less than $-R$, and have positive residues.
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- The poles of $f^{[m/n]}(z)$ are simple, real and less than $-R$, and have positive residues.

$$f^{[n-1/n]}(z) = \frac{(-z)^{n-1} q_{n-1}(-1/z)}{(-z)^n p_n(-1/z)} = \sum_{i=1}^{n} \frac{w_i}{1 + x_i z}.$$
Some more theory on Stieltjes functions.

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- Plus convergence results
Outline

1 Introduction

2 Theoretical results

3 Numerical results
Comparing the Lévy density

Figure: The graph of $x\pi(x)$ (black curve) and $x\pi^{[n/n]}(x)$, where $\pi(x) = \exp(-x)/x$ is the Lévy density of the Gamma subordinator, and $\pi^{[n/n]}(x)$ is the Lévy density corresponding to $\psi^{[n/n]}(z)$ Padé approximation. Blue, green and red curves correspond to $n \in \{5, 10, 20\}$. 
Comparing the Lévy density

**Figure:** The graph of $x\pi(x)$ (black curve) and $x\pi^{n/n}(x)$, where $\pi(x) = \exp(-x)/x$ is the Lévy density of the Gamma subordinator, and $\pi^{n/n}(x)$ is the Lévy density corresponding to $\psi^{n/n}(z)$ Padé approximation. Blue, green and red curves correspond to $n \in \{5, 10, 20\}$. 
Comparing the CDF of $X_2$

<table>
<thead>
<tr>
<th>$\epsilon_{n,k}(2)$</th>
<th>$k = 0$</th>
<th>$k = 1$</th>
<th>$k = 2$</th>
</tr>
</thead>
<tbody>
<tr>
<td>$n = 5$</td>
<td>$3.3e-4$</td>
<td>$3.2e-4$</td>
<td>$5.4e-4$</td>
</tr>
<tr>
<td>$n = 10$</td>
<td>$2.6e-5$</td>
<td>$2.8e-5$</td>
<td>$5.6e-5$</td>
</tr>
<tr>
<td>$n = 15$</td>
<td>$5.4e-6$</td>
<td>$6.4e-6$</td>
<td>$1.3e-5$</td>
</tr>
<tr>
<td>$n = 20$</td>
<td>$1.8e-6$</td>
<td>$2.1e-6$</td>
<td>$4.6e-6$</td>
</tr>
</tbody>
</table>

**Table:** The values of $\epsilon_{n,k}(t) := \max_{x \geq 0} |\mathbb{P}(X_t \leq x) - \mathbb{P}(X_{(n,k)}^t \leq x)|$, where $X$ is the Gamma process with $\psi(z) = -\ln(1 - z)$ and the process $X_{(n,k)}$ has Laplace exponent $\psi^{[n+k/n]}$. 
Math Finance applications

We will work with the following two processes: the VG process $V$ defined by the Laplace exponent

$$\psi(z) = \mu z - \frac{1}{\nu} \log \left(1 - \frac{z}{a}\right) - \frac{1}{\nu} \log \left(1 + \frac{z}{\hat{a}}\right),$$

and parameters

$$(a, \hat{a}, \nu) = (21.8735, 56.4414, 0.20),$$

and the CGMY process $Z$ defined by the Laplace exponent

$$\psi(z) = \mu z + C \Gamma(-Y) \left[(M - z)^Y - M^Y + (G + z)^Y - G^Y\right],$$

and parameters

$$(C, G, M, Y) = (1, 8.8, 14.5, 1.2).$$
## European call option

<table>
<thead>
<tr>
<th></th>
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</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>-1.58e-2</td>
<td>9.12e-2</td>
<td>7.02e-3</td>
<td>-3.02e-2</td>
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<td>-1.37e-9</td>
<td>-3.31e-9</td>
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**Table:** The error in computing the price of the European call option for the VG $V$-model. Initial stock price is $S_0 = 100$, strike price $K = 100$, maturity $T = 0.25$ and interest rate $r = 0.04$. The benchmark price is 2.5002779303.
European call option

<table>
<thead>
<tr>
<th></th>
<th></th>
<th></th>
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</tr>
</thead>
<tbody>
<tr>
<td>$N = 1$</td>
<td>-2.75e-2</td>
<td>1.93e-2</td>
<td>-3.72e-3</td>
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<tr>
<td>$N = 2$</td>
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<td>9.5e-5</td>
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<tr>
<td>$N = 3$</td>
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<td>-1.48e-5</td>
<td>-2.54e-7</td>
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<tr>
<td>$N = 4$</td>
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<td>6.41e-7</td>
<td>-1.55e-7</td>
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<td>1.14e-9</td>
<td>5.58e-9</td>
<td>6.95e-9</td>
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Table: The error in computing the price of the European call option for the CGMY $Z$-model. The benchmark price is 11.9207826467.
Asian option

Asian call option

\[ C(S_0, K, T) := e^{-rT} \mathbb{E}\left[ \left( \int_0^T S_u du - K \right)^+ \right] \]

We set the parameters \( S_0 = 100, r = 0.03, T = 1, K = 90 \) for the VG process and \( K = 110 \) for the CGMY process.

\[
\begin{array}{|c|c|c|c|c|}
\hline
N & \text{two-sided} & \text{one-sided} & \text{one-sided} & \text{one-sided} \\
 & \text{[2N + 1/2N]} & \text{[N/N]} & \text{[N + 1/N]} & \text{[N + 2/N]} \\
\hline
1 & -1.87e-3 & 1.01e-3 & -1.82e-3 & 9.88e-4 \\
2 & 9.49e-5 & 2.89e-4 & -6.33e-5 & 3.27e-5 \\
3 & 1.30e-6 & 8.85e-6 & -4.24e-6 & 3.99e-6 \\
4 & -2.83e-6 & 1.07e-6 & -1.36e-6 & 3.16e-7 \\
5 & -1.11e-7 & -2.48e-8 & -5.91e-7 & -3.81e-7 \\
\hline
\end{array}
\]

Table: The error in computing the price of the Asian option for the VG \( V \)-model. The benchmark price is 11.188589 (calculated using the \([91/90]\) two-sided approximation).
Asian option

<table>
<thead>
<tr>
<th>N</th>
<th>two-sided ([2N + 1/2N])</th>
<th>one-sided ([N + 1/N])</th>
<th>one-sided ([N + 2/N])</th>
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<td>7.42e-4</td>
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<td>2</td>
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<td>-1.21e-7</td>
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<tr>
<td>5</td>
<td>-5.26e-7</td>
<td>-2.47e-7</td>
<td>-2.49e-7</td>
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</tbody>
</table>

Table: The error in computing the price of the Asian option for the CGMY Z-model. The benchmark price is 9.959300 (calculated using the [91/90] two-sided approximation).
Barrier option

Down-and-out barrier put option:

\[ D(S_0, K, B, T) := e^{-rT}E \left[ (K - S_T)^+ 1\{S_t>B \text{ for } 0\leq t\leq T\} \right], \]

We calculate barrier option prices for the process \( V \), for four values \( S_0 \in \{81, 91, 101, 111\} \) and with other parameters given by \( K = 100, B = 80, r = 0.04879 \) and \( T = 0.5 \)

<table>
<thead>
<tr>
<th></th>
<th>( S_0 = 81 )</th>
<th>( S_0 = 91 )</th>
<th>( S_0 = 101 )</th>
<th>( S_0 = 111 )</th>
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</thead>
<tbody>
<tr>
<td>Benchmark</td>
<td>3.39880</td>
<td>7.38668</td>
<td>1.40351</td>
<td>0.04280</td>
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<td>( N = 2 )</td>
<td>3.44551</td>
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<td>3.40209</td>
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<td>3.39910</td>
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<td>0.04258</td>
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<tr>
<td>( N = 8 )</td>
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<td>7.38936</td>
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<td>( N = 10 )</td>
<td>3.39853</td>
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<td>1.40332</td>
<td>0.04258</td>
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</tbody>
</table>

**Table:** Barrier option prices calculated for the VG process \( V \)-model. Benchmark prices obtained from “Fast and accurate pricing of barrier options under Lévy processes” by Kudryavtsev and Levendorskiï.
References:

- D. Hackmann and A. Kuznetsov (2014)

  Padé approximation of Stieltjes series.

  *Padé Approximants*, volume 1.

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