# On log-normal convolutions: An analytical-numerical method with applications to economic capital determination

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**Abstract**

We put forward an efficient algorithm for approximating the sums of independent and log-normally distributed random variables. Namely, by merging tools from probability theory and numerical analysis, we are able to compute the cumulative distribution functions of the just-mentioned sums with any desired precision. Importantly, our algorithm is fast and can tackle equally well sums with just a few or thousands of summands. We illustrate the effectiveness of the new method in the contexts of the individual and collective risk models, aggregate economic capital determination, and economic capital allocation.

*Keywords*: log-normal distribution, convolution, generalized gamma convolution, Padé approximation, individual risk model, collective risk model, economic capital

*JEL Classification*: C02, C46, C63

## 1 Introduction

The log-normal distribution has been found appropriate for modelling losses originating from a great variety of non-life insurance risks (e.g., Mikosch, 2009; Klugman et al., 2012). More specifically, Kleiber and Kotz (2003) mention applications in property, fire, hurricane, and motor insurances, to name just a few (also, e.g., Dropkin, 1964; Bickerstaff, 1972; O'Neill and Wells, 1972). Furthermore the standard

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formula of the European Insurance and Occupational Pensions Authority explicitely assumes the log-normality of insurers' losses (EIOPA-14-322, 2014). Finally, the role of the log-normal distribution is at least as profound in finance, where it serves as the canonical model describing stock price returns (e.g., Sprenkle, 1964; Milevsky and Posner, 1998). For a multitude of applications of the log-normal distribution in other areas we refer to Limpert et al. (2001), and references therein.

Recall that the random variable (r.v.) $X_{\mu,\sigma}$ is said to be distributed log-normally with parameters $\mu \in (-\infty, \infty)$ and $\sigma > 0$, succinctly $X_{\mu,\sigma} \sim LN(\mu, \sigma^2)$, if its probability density function (p.d.f.) is given by

$$f(x) = \frac{1}{x\sqrt{2\pi\sigma^2}} \exp\left(-\frac{1}{2}\left(\frac{\ln(x)-\mu}{\sigma}\right)^2\right) \quad \text{for } x > 0. \tag{1}$$

Although $X_{\mu,\sigma}$ is merely a simple transform of the well-understood Gaussian r.v., it has been puzzling researchers for decades. For instance, an explicit expression for the Laplace transform of (1) has eluded mathematicians thus far, and the existing series representations are rather cumbersome (Leipnik, 1991).

In this paper we are interested in a related problem of *log-normal convolutions*. In fact, the convolutions of log-normal distributions have drawn considerable attention of researchers and practitioners due to the fundamental importance of log-normal sums in engineering, biology, ecology, and economics, as well as in actuarial science and finance. It is not surprising therefore that the existing contributions on the topic are abundant and span all of the just-mentioned areas (e.g., Dufresne, 2008; Asmussen et al., 2011, for recent literature reviews).

The problem is admittedly very intricate, and no explicit solution is generally available. The existing methods to approximate the log-normal convolutions can be associated with the following three main threads, or their variants: (i) moment matching method and its modifications, (ii) series representations of the Laplace transform, and (iii) asymptotic results.

The moment matching method is akin to the idea of approximating the convolutions of log-normal distributions by means of (other) distributions. It seems that for the first time this method was documented in Fenton (1960), who employs the log-normal distribution with the first and second moments being equal to the first and second moments of the desired log-normal sum (e.g., Beaulieu et al., 1995; Milevsky and Posner, 1998). The idea is further refined in Chen et al. (2008) and Zhang and Song (2008), who utilize the four-parameter Pearson IV family of distributions to approximate the distribution of log-normal sums. An alternative - analytical - way to approximate the convolutions of log-normal distributions is by inverting the corresponding Laplace transform, which is to this end expanded into a series. In this respect, Holgate (1989) discusses several techniques for approximating the characteristic function of the log-normal distribution using the re-summation of divergent series, and also provides asymptotic approximations, and Leipnik (1991) derives a convergent series representation for the characteristic function. Last but not least, asymptotic approximations of the convolutions of log-normal distributions are derived in Asmussen and Rojas-Nandayapa (2008) (see also Dhaene et al., 2008). Methods that do not

immediately fit within the reseacrh directions above also exist (e.g., Vanduffel et al., 2008).

Unfortunately, the existing approaches may deliver inaccurate results. This is particularly so for large values of the $\sigma$ parameter, and when a small number of log-normally distributed summands are considered. The reasons are that in the former case the distribution of $X_{\mu,\sigma}$ would have heavier tails, and in the latter case the Central Limit Theorem would not apply (Asmussen et al., 2011). The solution that we propose in this paper is distinct in that it hinges on a synthesis of several tools from probability theory and numerical analysis. More specifically, we construct an approximant r.v., such that the Laplace transform of its distribution converges uniformly and exponentially fast to the Laplace transform of the distribution of $X_{\mu,\sigma}$. The approximating Laplace transform of the desired convolution then follows immediately, and the cumulative distribution function (c.d.f.) of the convolution is obtained via routine inversion techniques. The proposed method performs equally well in the tail region of the distribution of the sum, and in its 'body', it is quick when tackling numerous summands of varying tail thickness, allows for any level of accuracy, and, last but not least, it can be generalized to compute the c.d.f.'s of the sums of independent and not-necessarily log-normally distributed r.v.'s.

We illustrate the efficiency of our approach with a few examples borrowed from the context of economic capital determination and allocation within the individual and collective risk models. We recall in this respect that, for mutually independent r.v.'s $X_1, X_2, \ldots$, the r.v.

$$S_N := X_1 + \cdots + X_N$$

is called an Individual Risk Model (IRM) if $N$ is a positive deterministic integer, and it is called a Collective Risk Model (CRM) if $N$ is random; in the latter case, the r.v.'s $X_1, X_2, \ldots$ are also assumed to be identically distributed (i.i.d.). (In what follows, we use the symbol ":=" when we feel it is necessary to emphasize that an equality is by definition.)

The aforementioned choice of examples is obviously not ad hoc. Indeed, besides the clear link to the notion of convolutions, the individual and, also, collective risk models have been taught to actuarial students for many years now, and the two models have manifested ubiquitously in both theoretical and practical loss modelling (e.g., Kaas et al., 2008; Klugman et al., 2012). In addition, from the point of view of the modern insurance regulation (e.g., Solvency II, and equivalents), it is essential to evaluate the economic capital required for supporting the aggregate risk r.v. $S_N$ (herein we use the notions "risk" and "loss" interchangeably). In this paper we touch on all of the above. Namely, we assume that $X_1, X_2, \ldots$ are independent and log-normally distributed r.v.'s and compute the Value-at-Risk (VaR)

$$\mathrm{VaR}_q[S_N] := \inf\{s \in \mathbb{R} : \mathbb{P}[S_N \leq s] \geq q\} \tag{2}$$

and the Conditional Tail Expectation (CTE)

$$\mathrm{CTE}_q[S_N] := \mathbb{E}[S_N |\ S_N > \mathrm{VaR}_q[S_N]] \ \text{ for } S_N \text{ with finite mean} \tag{3}$$

risk measures, where $q \in (0,\ 1)$ is the prudence parameter, and $N$ is deterministic or random. The VaR and the CTE risk measures do not require advertisement, as their popularity in insurance and banking is immense (Artzner et al., 1999; McNeil et al., 2005; Denuit et al., 2006). For the situations when the variability of the tail risk is of interest, we compute the Tail Variance (TV) risk measure

$$\mathrm{TV}_q[S_N] := \mathbf{Var}[S_N|\ S_N > \mathrm{VaR}_q[S_N]] \ \text{ for } S_N \text{ with finite variance} \tag{4}$$

(Furman and Landsman, 2006). Furthermore, we compute the economic capital allocations that correspond to risk measures (3) and (4) (Furman and Zitikis, 2008; Dhaene et al., 2012, and references therein) when the r.v.'s $X_1, \ldots, X_N$ are distributed log-normally, and $N$ is deterministic or random.

The rest of the paper is organized as follows. In Section 2 we describe our method, formulate and prove the main results. Further, in Section 3 we build the required approximating scheme, and then in Sections 4, 5 and 6 elucidate it with examples. In Section 7 we discuss the computation time of our algorithm, and Section 8 provides concluding remarks. Some well-known but worthy to mention details about the numerical inversion of the Laplace transform are relegated to Appendix A.

## 2 The analytical basis for the proposed method

From now and on, we often set $\mu = 0$ and thus work with a unit scale log-normally distributed r.v. $X_\sigma \sim LN(0, \sigma^2),\ \sigma > 0$. For simplicity of notation, we write $X_\sigma \sim LN(\sigma^2)$.

**Remark 1.** In this paper we approximate the Laplace transform $\phi_\sigma(z) := \mathbf{E}\left[\exp(-zX_\sigma)\right]$ with the help of a simpler function $\widetilde{\phi}_\sigma(z)$ for $\mathrm{Re}(z) \geq 0$. Obviously, once the approximation $\widetilde{\phi}$ has been constructed, it can be used to approximate the Laplace transform of the more general three parameter log-normal p.d.f. (e.g., O'Neill and Wells, 1972)

$$f(x) = \frac{1}{x\sqrt{2\pi\sigma^2}} \exp\left(-\frac{1}{2}\left(\frac{\ln(x-\tau)-\mu}{\sigma}\right)^2\right) \quad \text{for } x > \tau, \tag{5}$$

where $e^\mu > 0$ and $\sigma > 0$ are, respectively, the scale and shape parameters, because the Laplace transform of (5) is given by $e^{-z\tau}\phi_\sigma\left(ze^\mu\right)$ for $\mathrm{Re}(z) \geq 0$.

It often happens in the mathematical sciences that generalizing an object highlights its essence and in this way helps to understand it better. In our context, the answer to the question as to what the distribution of the sum of independent and log-normally distributed r.v.'s is would be rather unclear, unless a significantly more encompassing class of Generalized Gamma Convolutions (GGC's) - of which the log-normal distribution is a member - is considered.

More formally, it is known that the distribution of $X_\sigma \sim LN(\sigma^2)$ is a limiting distribution of a sequence of convolutions of gamma distributions, and it is thus infinitely divisible (Thorin, 1977; Bondesson, 2002). This motivates the introduction and study of the class of GGC's as follows.

**Definition 1** (Thorin (1977); Bondesson (1992))**.** The distribution on $[0,\ \infty)$ of the r.v. $X$ is a GGC if its Laplace transform is

$$\phi(z) := \mathbb{E}[\exp(-zX)] = \exp\Big( -az - \int_0^\infty \ln(1+z/t)U(\mathrm{d}t)\Big) \text{ for } \mathrm{Re}(z) \geq 0, \tag{6}$$

where $a \in [0,\ \infty)$ is a constant, and $U(\mathrm{d}t)$ is a positive Radon measure, also called *Thorin* measure, which must satisfy

$$\int_0^\infty \min(|\ln(t)|, 1/t)U(dt) < \infty.$$

**Remark 2.** Set $a = 0$ in (6) for convenience and without loss of generality, and notice that if $U(dt)$ is a discrete measure, then (6) is the Laplace transform of a finite convolution of gamma distributions, thus motivating the name *GGC*.

We now briefly explain the main idea behind our approach. Let $X$ be a random variable with a distribution in the class of GGC's, and let $\Gamma_i \sim Ga(\alpha_i, \beta_i)$ denote independent gamma distributed r.v.'s with shape parameters $\alpha_i > 0$ and rate parameters $\beta_i > 0$. The distribution of $X$ can be approximated arbitrarily well by distributions of finite sums of the form

$$X_n := \sum_{i=1}^n \Gamma_i. \tag{7}$$

In other words, for each $n \in \mathbb{N}$, there exist parameters $\alpha_i = \alpha_i(n)$ and $\beta_i = \beta_i(n)$, $1 \leq i \leq n$, such that $X_n \to X$ in distribution. The two (closely related) questions that we would like to answer are the following:

(i) Given the distribution of $X$, is there a practical way to compute the parameters $\alpha_i$ and $\beta_i$ of the distribution of the approximating r.v. (7)?

(ii) Can we choose the parameters $\alpha_i$ and $\beta_i$ in some sort of 'optimal way', so that the distribution of $X_n$ is as 'close' as possible to the distribution of $X$?

Note that approximating the distribution of $X$ with the help of the distribution of r.v. (7) is equivalent to approximating the Laplace transform $\phi(z) = \mathbb{E}[\exp(-zX)]$ by functions of the form

$$\phi_n(z) := \mathbb{E}[\exp(-zX_n)] = \prod_{i=1}^n (1+z/\beta_i)^{-\alpha_i} \text{ for } z > 0. \tag{8}$$

Thus our second question can be rephrased more formally in the following way: can we choose the parameters $\alpha_i$ and $\beta_i$ so that as $n \to +\infty$ the functions $\phi_n(z)$ converge to $\phi(z)$ as fast as possible in the half-plane $\mathrm{Re}(z) \geq 0$? In our next theorem we show that one can choose parameters $\alpha_i$ and $\beta_i$ that guarantee exponential rate of convergence, and we provide an explicit algorithm on how to compute these parameters later.

Before stating the next result, we need to present one definition. For $c > 0$ and a positive r.v. $X$ we define *the Escher transform* r.v. $X^{(c)}$ via the distribution function

$$\mathbb{P}(X^{(c)} \in \mathrm{d}x) = \frac{e^{-cx}}{\mathbb{E}[e^{-cX}]}\mathbb{P}(X \in \mathrm{d}x) \text{ for } x > 0.$$

Now we are ready to state our main result in this section.

**Theorem 1.** *Let the r.v. $X$ have a distribution in the class of GGC's, and assume that $X$ is not of the form in (7). Fix $z^* > 0$, then the following three statements hold:*

*(i) For any $n \in \mathbb{N}$, there exist positive numbers $\{\alpha_i\}_{1\leq i\leq n}$ and $\{\beta_i\}_{1\leq i\leq n}$ such that the r.v. $X_n$ defined in (7) satisfies*

$$\mathbb{E}[(X_n^{(z^*)})^k] = \mathbb{E}[(X^{(z^*)})^k] \tag{9}$$

*for $1 \leq k \leq 2n$;*

*(ii) The numbers $\{\alpha_i\}_{1\leq i\leq n}$ and $\{\beta_i\}_{1\leq i\leq n}$ are unique (up to permutation);*

*(iii) The functions $\phi_n(z) = \mathbb{E}[\exp(-zX_n)]$ converge to $\phi(z) = \mathbb{E}[\exp(-zX)]$ uniformly and exponentially fast on compact subsets of $\mathbb{C} \setminus (-\infty, 0]$. In particular, $X_n \xrightarrow{d} X$.*

**Remark 3.** Statement (i) in Theorem 1 warrants that the proposed approximating scheme is parsimonious. More specifically, in order to match the first $2n$ moments of the Esscher transforms of the distributions of the r.v.'s $X_n$ and $X$ we would need at least $2n$ parameters, thus our approximation is optimal in this sense.

**Remark 4.** In general one can not take $z^* = 0$ in Theorem 1. This is due to the fact that not all distributions are determined by their moments (for example, it is well-known that the log-normal distribution is not uniquely determined by its moments). Therefore the use of Escher transform is unavoidable in our method.

**Remark 5.** The parameter $n \in \mathbb{N}$ of the approximating Laplace transform regulates the accuracy of the approximation (the larger, the better). We keep the subscript when a confusion may arise as to what the degree of the approximation is (Sections 3 and 4), and we omit it - and write $\tilde{\phi}$ to simplify the notation - when such degree is fixed (the rest of the paper).

*Proof of Theorem 1.* Without loss of generality we may assume that $a = 0$ in (6). Let $U(\mathrm{d}t)$ be the Thorin measure as appears in (6). We define the two functions

$$\psi(z) = -\frac{\mathrm{d}}{\mathrm{d}z}\ln(\phi(z)) = \int_0^\infty \frac{U(\mathrm{d}t)}{t+z}, \tag{10}$$

and $\psi^*(z) = \psi(z + z^*)$ for $z > 0$. The function $\psi$ is analytic in $\mathbb{C} \setminus (-\infty, 0]$, thus $\psi^*$ is analytic in $\mathbb{C} \setminus (-\infty, -z^*]$. After the change of variables $w = 1/(t + z^*)$ in (10), the function $\psi^*$ can be written in the form

$$\psi^*(z) = \int_0^{1/z^*} \frac{\mu(\mathrm{d}w)}{1 + wz} \tag{11}$$

for a positive measure $\mu(\mathrm{d}w)$ on $(0, 1/z^*)$, which is simply the pushforward of $U(\mathrm{d}t)$ using the function $w(t) = 1/(t + z^*)$. Integral representation (11) tells us that $\psi^*(z)$ is a Stieltjes function, and it is well-known (e.g., Baker and Graves-Morris, 1996) that Stieltjes functions can be approximated very well by certain rational functions, called *Padé approximants*. This is the central idea behind the proof of Theorem 1.

First of all, we note that the function $\psi^*$ is analytic in the disk $|z| < z^*$, thus it can be expanded in Taylor series as follows

$$\psi^*(z) = \sum_{k \geq 0} s_k z^k, \tag{12}$$

where

$$s_k = \frac{\psi^{*(k)}(0)}{k!} = \frac{\psi^{(k)}(z^*)}{k!}. \tag{13}$$

The $[n - 1/n]$ Padé approximation to $\psi^*$ is a rational function of the form $P(z)/Q(z)$, where $P(z) = a_0 + a_1 z + a_2 z^2 + \cdots + a_{n-1} z^{n-1}$ and $Q(z) = 1 + b_1 z + b_2 z^2 + \cdots + b_n z^n$ are two polynomials that satisfy

$$\frac{P(z)}{Q(z)} - \psi^*(z) = O(z^{2n}) \text{ for } z \to 0. \tag{14}$$

In other words, the first $2n$ coefficients of the Taylor expansion of $P(z)/Q(z)$ at zero should match the corresponding $2n$ coefficients of $\psi^*$. According to Corollary 1 on page 164 in Baker and Graves-Morris (1996), due to the fact that $\psi^*$ is a Stieltjes function, the $[n - 1/n]$ Padé approximation to $\psi^*$ exists and is unique. Furthermore, Theorem 5.4.1 in Baker and Graves-Morris (1996) (also Theorems 2.2 and 3.1 in Allen et al., 1975) tells us that the denominator $Q(z)$ of the $[n - 1/n]$ Padé approximation has $n$ simple zeros $z_i$ that lie in $(-\infty, -z^*)$ and that the rational function $\psi_n^*(z) := P(z)/Q(z)$ can be written in the partial fraction form as

$$\psi_n^*(z) = \sum_{i=1}^{n} \frac{\alpha_i}{z - z_i}, \tag{15}$$

where $\alpha_i = P(z_i)/Q'(z_i) > 0$, for $i = 1, \ldots, n$. Finally, according to Theorem 5.4.4 in Baker and Graves-Morris (1996), as $n \to +\infty$ we have $\psi_n^*(z) \to \psi^*(z)$ exponentially fast on compact subsets of $\mathbb{C} \setminus (-\infty, -z^*]$.

Let us define $\beta_i = -z_i - z^*$ (note that $\beta_i > 0$). The above facts imply that as $n \to +\infty$ the functions

$$\phi_n(z) = \exp\left( - \int_0^z \psi_n^*(w - z^*) \mathrm{d}w \right) = \prod_{i=1}^{n} (1 + z/\beta_i)^{-\alpha_i}$$

converge to

$$\phi(z) = \exp\left(-\int_0^z \psi^*(w - z^*)\mathrm{d}w\right)$$

exponentially fast on compact subsets of $\mathbb{C} \setminus (-\infty, 0]$. This completes the proof of parts (i) and (iii) of the theorem.

Let us now prove part (ii). Assume that we have found numbers $\alpha_i$ and $\beta_i$ such that part (i) holds. Define $\phi_n(z)$ as in (8). Then the statement in part (i) implies that the rational function $\frac{\mathrm{d}}{\mathrm{d}z}\phi_n(z)$ is the $[n-1/n]$ Padé approximation to $\psi^*(z)$. Thus the uniqueness of this Padé approximation implies the uniqueness (up to permutation) of the coefficients $\alpha_i$ and $\beta_i$. This completes the proof of the theorem. □

# 3 The numerical basis for the proposed method

In this section we show how to compute numerically the coefficients $\alpha_i$ and $\beta_i$. The algorithm is quite straightforward, the only difficulty being that it requires the use of high-precision arithmetic.

## 3.1 Computing the coefficients $g_k$

The starting point for our algorithm is the numerical computation of the following numbers

$$g_k := \phi^{(k)}(z^*) = (-1)^k \mathbb{E}[X^k \exp(-z^* X)] \quad \text{for} \quad 0 \le k \le 2n. \tag{16}$$

We compute $g_k$ by numerical integration

$$g_k = (-1)^k \int_0^\infty x^k e^{-z^* x} f_X(x)\mathrm{d}x,$$

where $f_X$ is the p.d.f. of the r.v. $X$. To do this efficiently, we use the double-exponential quadrature of Takahasi and Mori (1974): we perform a change of variables $x = u(y) = \exp(y - \exp(-y))$, $y \in (-\infty, \infty)$, and approximate the resulting integral by a Riemann sum

$$I_h := h \sum_{m=-\infty}^{\infty} u(mh)^k e^{-z^* u(mh)} f_X(u(mh))(1 + e^{-mh})u(mh).$$

For nice enough functions $f_X(x)$, the rate of convergence of $I_h$ to $g_k$ is very fast (as $h \to 0^+$). For example, when we take $f_X$ to be a log-normal p.d.f. with $\mu = 0$ and $\sigma = 0.5$, we find that taking $h = 0.008$ and truncating the above infinite series to the range $-1000 < m < 1000$ allows us to compute the first forty coefficients $g_k$ with accuracy better than $1.0e - 300$, which is sufficient for our purposes.

### 3.2 Computing the coefficients $s_k$

Our next goal is to compute the coefficients $s_k$ defined in (13). We observe that the equation $\psi^*(z) = -\frac{\mathrm{d}}{\mathrm{d}z}\ln(\phi(z+z^*))$ can be rewritten as

$$\frac{d}{dz}\phi(z+z^*) = -\psi^*(z)\phi(z+z^*), \tag{17}$$

or, equivalently, as

$$\sum_{k\geq 0} g_{k+1}z^k/k! = -\left(\sum_{n\geq 0} s_n z^n\right) \times \left(\sum_{k\geq 0} g_k z^k/k!\right).$$

Comparing the constant term in the Taylor series in the left-hand side and the right-hand side of the above equation we find that $s_0 = -g_1/g_0$, and comparing the coefficients in front of $z^k$ we obtain the identity

$$s_k = -\frac{1}{g_0}\left(\frac{g_{k+1}}{k!} + \sum_{i=0}^{k-1} s_i \frac{g_{k-i}}{(k-i)!}\right), \tag{18}$$

which allows us to compute $s_k$ recursively for all $1 \leq k \leq 2n-1$.

Computing the coefficients $s_k$ via recursion (18) requires the use of high-precision arithmetic. The reason is that the numbers $g_k$ grow very fast in absolute value and have alternating sign, thus (18) leads to subtracting large numbers and the inevitable loss of precision.

### 3.3 Computing the coefficients $a_0, \ldots, a_{n-1}$ and $b_1, \ldots, b_n$

Next, given the values of $\{s_i\}_{0\leq i\leq 2n-1}$, we compute the numbers $a_i$ and $b_i$ that provide us the coefficients of the numerator/denominator in the Padé approximation $P(z)/Q(z)$. The procedure is as follows: solve the system of $n$ linear equations

$$\begin{bmatrix} s_0 & s_1 & s_2 & \cdots & s_{n-1} \\ s_1 & s_2 & s_3 & \cdots & s_n \\ s_2 & s_3 & s_4 & \cdots & s_{n+1} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ s_{n-1} & s_n & s_{n+1} & \cdots & s_{2n-2} \end{bmatrix} \begin{bmatrix} b_n \\ b_{n-1} \\ b_{n-2} \\ \vdots \\ b_1 \end{bmatrix} = - \begin{bmatrix} s_n \\ s_{n+1} \\ s_{n+2} \\ \vdots \\ s_{2n-1} \end{bmatrix} \tag{19}$$

to calculate $b_i$, $1 \leq i \leq n$. Then set $a_0 = s_0$ and compute

$$a_k = s_k + \sum_{i=1}^{k} b_i s_{k-i} \tag{20}$$

for $1 \leq k \leq n-1$. Solving the above system of linear equations again requires the use of high-precision arithmetic, since the matrix may be ill-conditioned.

To justify the above method, we rewrite condition (14) in the form

$$(a_0 + a_1 z + a_2 z^2 + \cdots + a_{n-1} z^{n-1}) - \left( \sum_{n \geq 0} s_n z^n \right) (1 + b_1 z + b_2 z^2 + \cdots + b_n z^n) = O(z^{2n}) \text{ for } z \to 0. \quad (21)$$

By comparing the coefficients of $z^k$ for $n \leq k \leq 2n-1$ we obtain system of equations (19), and formulas (20) follow by comparing the coefficients of $z^k$ for $0 \leq k \leq n-1$ in (21).

### 3.4 Computing the coefficients $\alpha_1, \ldots, \alpha_n$ and $\beta_1, \ldots, \beta_n$

In this last step of the algorithm, we define polynomials $Q(z) := 1 + b_1 z + b_2 z^2 + \cdots + b_n z^n$ and $P(z) := a_0 + a_1 z + a_2 z^2 + \cdots + a_{n-1} z^{n-1}$. We compute $\{z_i\}_{1 \leq i \leq n}$, which are the zeroes of the polynomial $Q(z)$ (we know that they are all simple and lie in $(-\infty, -z^*)$). This step also requires the use of high-precision arithmetic.

Finally, we compute

$$\beta_i = -z_i - z^* \quad \text{and} \quad \alpha_i = \frac{P(z_i)}{Q'(z_i)} \quad (22)$$

for $1 \leq i \leq n$. These are the desired rate and shape parameters of the approximant r.v. $X_n$ defined in (7). Alternatively, these give us the parameters of the Laplace transform $\phi_n$ defined in (8).

## 4 Performance of the approximation algorithm: A simple example

We briefly elucidate the accuracy of the approximation algorithm in the case of a single log-normally distributed r.v. $X_{0.83}$. A few notes are instrumental at the moment. First, the value of the shape parameter $\sigma = 0.83$ has been chosen in line with the empirical evidence reported in O'Neill and Wells (1972) in the context of the collision claim payments involving 30 – 40 year old drivers. Second, starting off with the approximation of a single log-normally distributed r.v. allows us to use its c.d.f. as a benchmark of the appropriateness of our approximation (in the case of the sum $S_N$, we have no explicit expressions to compare to). Last but not least, as we have already emphasized, the choice of the location and scale parameters being equal to 0 and 1, respectively is made for convenience only, and the inclusion of the general three parameter log-normal distribution (e.g., O'Neill and Wells, 1972) is straightforward.

At the outset, recall that the algorithm sketched in Section 3 has two free parameters, which are $n \in \mathbb{N}$ and $z^* > 0$. The former parameter has a simple intuitive interpretation, that is larger values of $n$ yield more accurate approximations. The impact of the latter parameter is harder to describe. We have run several numerical experiments in this respect, computing the maximum absolute difference between the approximating and explicit c.d.f.'s for a large range of values of $z^* > 0$, and our conclusions

are that the choice of $z^*$ does not seem to affect the accuracy of the approximation, unless extremely large or extremely small values are employed. The best accuracy is typically achieved for $z^* \in [0.3,\ 5]$, and the error does not change significantly for the values of $z^*$ in this interval. Thus for all our further computations we fix $z^* = 1$.

To summarize, we set $z^* = 1$ and evoke the approximation algorithm with $n \in \{10,\ 20,\ 30,\ 40\}$ and then compute the c.d.f. of the approximant r.v.'s $X_{10}$, $X_{20}$, $X_{30}$, and $X_{40}$ via the inversion of the corresponding approximating Laplace transforms $\phi_{10}$, $\phi_{20}$, $\phi_{30}$, and $\phi_{40}$ (see, Appendix A; we note in passing that as $\phi_n$ are products of the form in (8), the conditions required to deriving (45) are met). We choose large number of discretization points when computing the inverse Laplace transforms and ensure that the errors from this step are less than $1.0e-12$. Then we compare the approximating c.d.f.'s with the explicit c.d.f. The outcomes are depicted in Figure 1. Remarkably, the figure suggests that the approximation error in the right tail is only visible for $n = 10$, and the approximating c.d.f.'s are not visually distinguishable from the original c.d.f. even for this small value of $n$.

# 5 Aggregate economic capital determination and allocation: Individual Risk Model

Recall the individual risk model (Kaas et al., 2008; Klugman et al., 2012). Namely, for a sequence of - in our case log-normally distributed - risk r.v.'s $X_{\sigma_1}, \ldots, X_{\sigma_N}$, and a deterministic constant $N \in \mathbb{N}$, we have

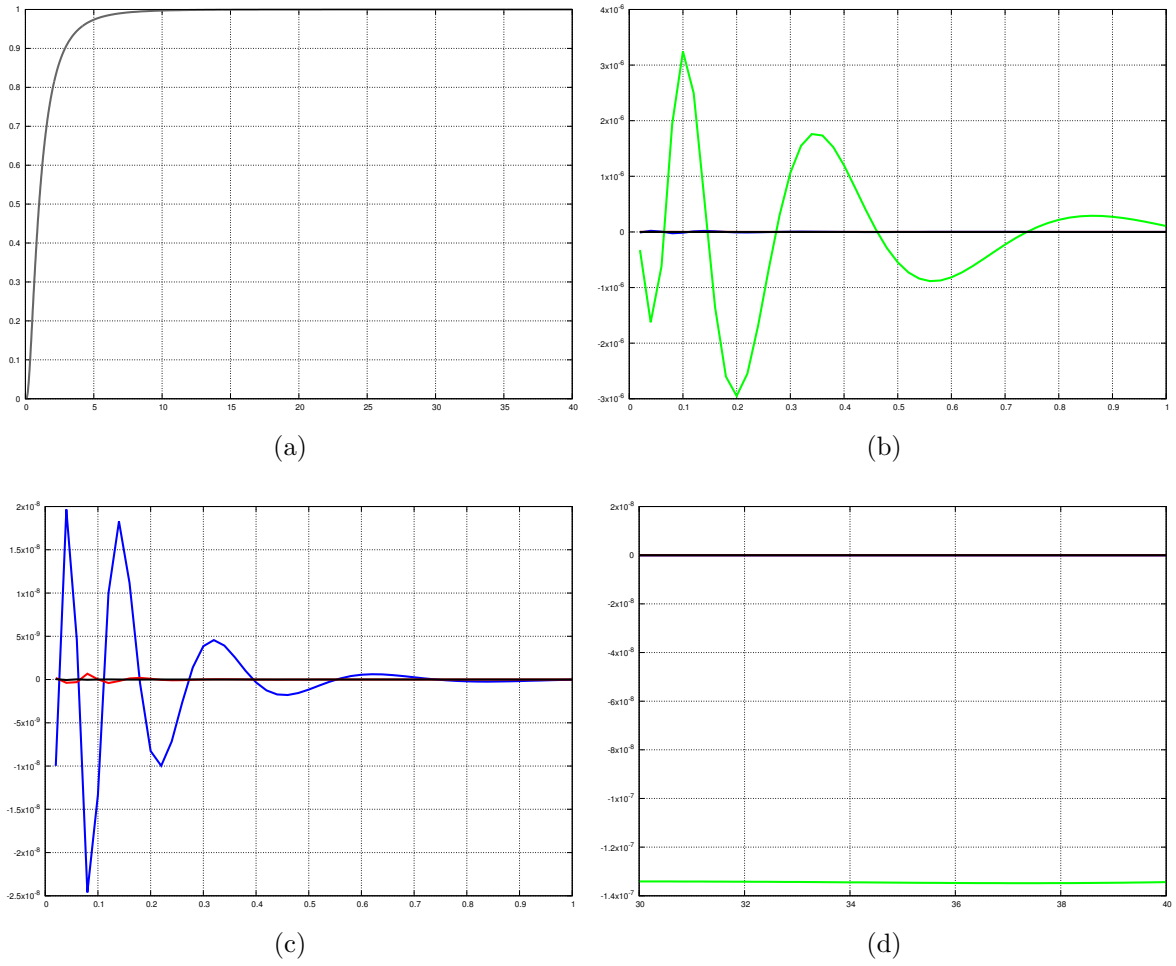
$$S_N = X_{\sigma_1} + \cdots + X_{\sigma_N} \tag{23}$$

interpreted as the aggregate risk of an insurer. Whether the summands $X_{\sigma_i}$, $i = 1, \ldots, N$ represent simple standalone risks, or, more cumbersome risks due to business lines of an insurer, they are assumed mutually independent within the IRM framework. However, unlike in the context of the collective risk model considered later on in Section 6, the risk r.v.'s $X_{\sigma_1}, \ldots, X_{\sigma_N}$ in (23) must not be identically distributed.

Theorem 1 allows to reduce the problem of computing the c.d.f.'s of r.v.'s (23) to a remarkably more tractable set-up of finite sums of independent gamma distributed r.v.'s. More specifically, we have the next corollary because the log-normal distribution is a GGC, and since $X_{\sigma_1}, \ldots, X_{\sigma_N}$ are independent r.v.'s.

**Corollary 1.** *Statements (i) and (iii) in Theorem 1 hold with the r.v. $X$ replaced with $S_N$.*

In the rest of the paper we aim at answering the following question: how much Economic Capital (EC) is required to support the risks $S_N$? To this end, let $H : \mathcal{X} \to [0, \infty]$ denote a regulatory risk



(a) (b)

(c) (d)

Figure 1: Whenever it applies, the colours: green, blue, red, and black correspond to the approximations of order $n = 10$, 20, 30, and 40, respectively. (a) The c.d.f. $F$ of $X_{0.83}$ compared with the c.d.f.'s $F_n$ of the approximant r.v.'s $X_n$. (b) The errors $F(x) - F_n(x)$ in the left tail. (c) Same as in panel (b) but omitting the $n = 10$ (green) approximation. (d) The errors $F(x) - F_n(x)$ in the right tail.

measure that maps risk r.v.'s in the set of actuarial risks $\mathcal{X}$ to EC's in the extended non-negative half-line. Nowadays the determination of the aggregate EC - $H[S_N]$ - is a compulsory task for insurers (e.g., Solvency II, Swiss Solvency Test). One of the most popular risk measures employed for this purpose is the conditional tail expectation

$$\mathrm{CTE}_q[S_N] = \mathbb{E}[S_N|\ S_N > \mathrm{VaR}_q[S_N]], \tag{24}$$

where $\mathrm{VaR}_q[S_N]$ is the Value-at-Risk, and $q \in [0,\ 1)$ is the prudence parameter set by the regulations. We note in passing that - for risk r.v.'s with continuous c.d.f.'s - the CTE risk measure coincides with the Expected Shortfall (ES) risk measure, and so it is coherent in the sense of Artzner et al. (1999) (also, Hürlimann, 2003; McNeil et al., 2005, Lemma 2.16), and belongs to the class of distorted (Wang, 1996) and weighted (Furman and Zitikis, 2008) risk measures.

If the variability along the right tail is of a concern, the modified Tail Variance (mTV) risk measure may become useful Furman and Landsman (2006) (also, Jiang et al., 2016, for a recent application). The mTV takes into account both the magnitude and the variability of the tail risk, and for the aggregate risk $S_N$ it is defined as

$$\mathrm{mTV}_q[S_N] = \mathbb{E}[S_N|\ S_N > \mathrm{VaR}_q[S_N]] + \frac{\mathbf{Var}[S_N|\ S_N > \mathrm{VaR}_q[S_N]}{\mathbb{E}[S_N|\ S_N > \mathrm{VaR}_q[S_N]]}, \tag{25}$$

where $q \in [0,\ 1)$ is the prudence parameter.

Our ultimate goal herein is to approximate (24) and (25). To this end, we have to find analytical expressions for (24) and (25) that involve Laplace transforms of the convolutions of log-normally distributed r.v.'s. We can then develop the desired approximations by substituting the aforementioned Laplace transforms with the approximating ones obtained with the help of Theorem 1. This requires a few auxiliary tools.

**Definition 2** (e.g., Patil and Rao (1978))**.** Let the r.v. $X$ have c.d.f. $F$ on $[0,\ \infty)$, and such that $\mathbb{E}[X^m] < \infty$ for $m \in \mathbb{N}$. Then the $m$-th order size-biased variant of $X$, succinctly $X^{*(m)}$, has the c.d.f.

$$F_{X^{*(m)}}(x) = \frac{\mathbb{E}[X^m \mathbf{1}\{X \le x\}]}{\mathbb{E}[X^m]} \text{ for } x \ge 0. \tag{26}$$

It is clear from the definition that the p.d.f. and the Laplace transform of $X^{*(m)}$ are, respectively,

$$f_{X^{*(m)}}(x) = \frac{x^m f(x)}{\mathbb{E}[X^m]} \text{ for } x > 0 \tag{27}$$

and

$$\phi_{X^{*(m)}}(z) := \mathbb{E}[\exp(-zX^{*(m)})] = \frac{\mathbb{E}[X^m e^{-zX}]}{\mathbb{E}[X^m]} \text{ for } \mathrm{Re}(z) \ge 0,\ m \in \mathbb{N}. \tag{28}$$

For $m = 1$, we simplify the notation and write $X^*$, $F_{X^*}$, $f_{X^*}$ and $\phi_{X^*}$ for the size-biased variant of $X$, its c.d.f., p.d.f. and Laplace transform.

When both the original and the size-biased c.d.f.'s belong to the same family of c.d.f.'s, we say that the distribution is closed under size-biasing of order $m \in \mathbb{N}$. In Patil and Rao (1978) it is observed that the log-normal distribution is closed under size-biasing of order one. That is, for $X_\sigma \sim LN(\sigma^2)$, we have $X_\sigma^* \sim LN(\sigma^2, \sigma^2)$, and as a result $\phi_{X_\sigma^*}(z) = \phi_{X_\sigma}(ze^{\sigma^2})$, $\mathrm{Re}(z) \geq 0$. We further show that the log-normal distribution is in fact closed under size-biasing of any order $m \in \mathbb{N}$. Moreover, we show that the size-biased variant of order $m$ of log-normal convolutions admits a finite mixture representation.

**Theorem 2.** *Let $X_\sigma \sim LN(\sigma^2)$, $\sigma > 0$ be a log-normally distributed r.v. with the corresponding Laplace transform $\phi_{X_\sigma}$. Also, let $X_{\sigma_j} \sim LN(\sigma_j)$ be mutually independent and log-normally distributed r.v.'s with shape parameters $\sigma_j > 0$ and the corresponding Laplace transforms $\phi_{X_{\sigma_j}}$, $j = 1, \ldots, N$. Finally, let $S_N = \sum_{j=1}^N X_{\sigma_j}$ as before. Then the following assertions hold:*

*(1) For any $m \in \mathbb{N}$ and $c = \exp(m\sigma^2)$, we have*

$$\phi_{X_\sigma^{*(m)}}(z) = \phi_{cX_\sigma}(z) \text{ for } \mathrm{Re}(z) \geq 0; \tag{29}$$

*(2) For any $m \in \mathbb{N}$ and $c_j = \exp(d_j\sigma_j^2)$, $j = 1, \ldots, N$, we have that the Laplace transform of the size-biased variant of order $m$ of the sum $S_N$ is the following weighted average of Laplace transforms*

$$\phi_{S_N^{*(m)}}(z) = \frac{1}{\mathbb{E}[S^m]} \sum_{d_1+\cdots+d_N=m} \binom{m}{d_1, \ldots, d_N} \prod_{j=1}^N \mathbb{E}\left[X_j^{d_j}\right] \phi_{c_1X_1+\cdots+c_NX_N}(z) \text{ for } \mathrm{Re}(z) \geq 0. \tag{30}$$

*Proof.* It is not difficult to check that, for $m \in \mathbb{N}$, we have $X_{\sigma_j}^{*(m)} \sim LN(m\sigma^2, \sigma^2)$, which proves (29). To confirm (30), we have the following string of equations

$$\begin{aligned}
\phi_{S_N^{*(m)}}(z) &= \frac{\mathbb{E}[S^m e^{-zS}]}{\mathbb{E}[S^m]} = \frac{1}{\mathbb{E}[S^m]}\mathbb{E}\left[\left(\sum_{j=1}^N X_j\right)^m e^{-zS}\right] \\
&= \frac{1}{\mathbb{E}[S^m]} \sum_{d_1+\cdots+d_N=m} \binom{m}{d_1, \ldots, d_N} \mathbb{E}\left[\prod_{j=1}^N X_j^{d_j} e^{-zX_j}\right] \\
&= \frac{1}{\mathbb{E}[S^m]} \sum_{d_1+\cdots+d_N=m} \binom{m}{d_1, \ldots, d_N} \prod_{j=1}^N \mathbb{E}\left[X_j^{d_j}\right] \prod_{j=1}^N \frac{\mathbb{E}\left[X_j^{d_j} e^{-zX_j}\right]}{\mathbb{E}[X_j^{d_j}]} \\
&= \frac{1}{\mathbb{E}[S^m]} \sum_{d_1+\cdots+d_N=m} \binom{m}{d_1, \ldots, d_N} \prod_{j=1}^N \mathbb{E}\left[X_j^{d_j}\right] \phi_{X_1^{*(d_1)}+\cdots+X_N^{*(d_N)}}(z) \\
&= \frac{1}{\mathbb{E}[S^m]} \sum_{d_1+\cdots+d_N=m} \binom{m}{d_1, \ldots, d_N} \prod_{j=1}^N \mathbb{E}\left[X_j^{d_j}\right] \phi_{c_1X_1+\cdots+c_NX_N}(z),
\end{aligned}$$

which completes the proof. □

Theorem 2 implies that, for any $m \in \mathbb{N}$, we can approximate the Laplace transforms of the p.d.f.'s of $X_{\sigma_j}^{*(m)}$ and $S_N^{*(m)}$ using Theorem 1, and this is precisely what we need latter on in this section. We note in passing that a particularly simple special case of (30) occurs for $m = 1$. Then we readily have, for $c_j = \exp(\sigma_j^2)$,

$$\phi_{S_N^*}(z) = \sum_{j=1}^{N} \frac{\mathbb{E}[X_j]}{\mathbb{E}[S_N]} \phi_{c_j X_{\sigma_j} + \sum_{i=1, i\neq j}^{N} X_{\sigma_i}}(z) \text{ for } \mathrm{Re}(z) \geq 0 \tag{31}$$

(e.g., Furman and Landsman, 2005, Proposition 1).

We are now ready to compute the approximations for (24) (Example 1) and (25) (Example 2). To this end, we choose $N = 3$ and assume that the constituents of (23) represent log-normally distributed risks due to three business lines of an insurer. We then derive the desired approximations using the algorithm described in Section 3 with $n = 20$. Since the order of the approximation does not vary any more, we use the "tilde" notation for all approximating quantities; e.g., $\widetilde{\phi}$, $\widetilde{\mathrm{CTE}}_q$ and $\widetilde{\mathrm{mTV}}_q$ denote, respectively, the approximating Laplace transform, CTE and mTV risk measures.

Let $p_i = \exp(\sigma_i^2/2),\ i = 1, \ldots, N$, and $p_+ = \sum_{i=1}^{N} p_i$.

**Example 1.** We start with the general IRM (see, (23)) with log-normally distributed constituents. Let $e(s) := \mathbb{E}[S_N \mathbf{1}\{S_N > s\}],\ s \geq 0$. Then - the proof is at the end of this section - we have

$$e(s) = \mathbb{E}[S_N] \mathcal{L}^{-1} \left\{ \frac{1 - \phi_{S_N^*}(z)}{z} \right\} (s) \text{ for } s \geq 0. \tag{32}$$

Consequently, for $q \in [0,\ 1)$, we obtain with the help of Theorem 2,

$$\begin{aligned} \mathrm{CTE}_q[S_N] &= \frac{p_+}{1-q} \mathcal{L}^{-1} \left\{ \frac{1 - \phi_{S_N^*}(z)}{z} \right\} (\mathrm{VaR}_q[S_N]) \\ &= \frac{p_+}{1-q} \mathcal{L}^{-1} \left\{ \frac{1}{z} \left( 1 - \sum_{i=1}^{N} \frac{p_i}{p_+} \phi_{c_i X_{\sigma_i} + \sum_{j=1, j\neq i}^{N} X_{\sigma_j}}(z) \right) \right\} (\mathrm{VaR}_q[S_N]). \end{aligned}$$

Therefore, for $N = 3$, $c_i = \exp(\sigma_i^2)$, and $q \in [0,\ 1)$, we have the following approximation

$$\mathrm{CTE}_q[S_3] \approx \frac{p_+}{1-q} \mathcal{L}^{-1} \left\{ \frac{1}{z} \left( 1 - \sum_{i=1}^{3} \frac{p_i}{p_+} \widetilde{\phi}_{c_i X_{\sigma_i} + \sum_{j=1, j\neq i}^{3} X_{\sigma_j}}(z) \right) \right\} (\widetilde{\mathrm{VaR}}_q[S_3]) = \widetilde{\mathrm{CTE}}_q[S_3], \tag{33}$$

where $\widetilde{\phi}$ and $\widetilde{\mathrm{VaR}}_q$ are obtained with the help of Theorem 1. This establishes the desired approximation.

To illustrate, assume that the r.v.'s $X_{\sigma_1}$, $X_{\sigma_2}$ and $X_{\sigma_3}$ are distributed as $X_{0.83} \sim LN(0.83)$. We depict (33) as well as the CTE of the comonotonic sum $S_3^c := 3X_{0.83}$ in Figure 2a. We note in passing that the two are the lower and the upper bounds, respectively, in the Fréchet set of all joint c.d.f.'s with fixed $LN(0.83)$ margins and varying positive cumulative dependence (Denuit et al., 2001). We compare our approach with the outcomes due to the Monte Carlo (MC) simulation method (50 MC simulations with $10^6$ random samples) in Figure 2b (see, also, Figures 2c); the superiority of our method is evident. This completes Example 1.

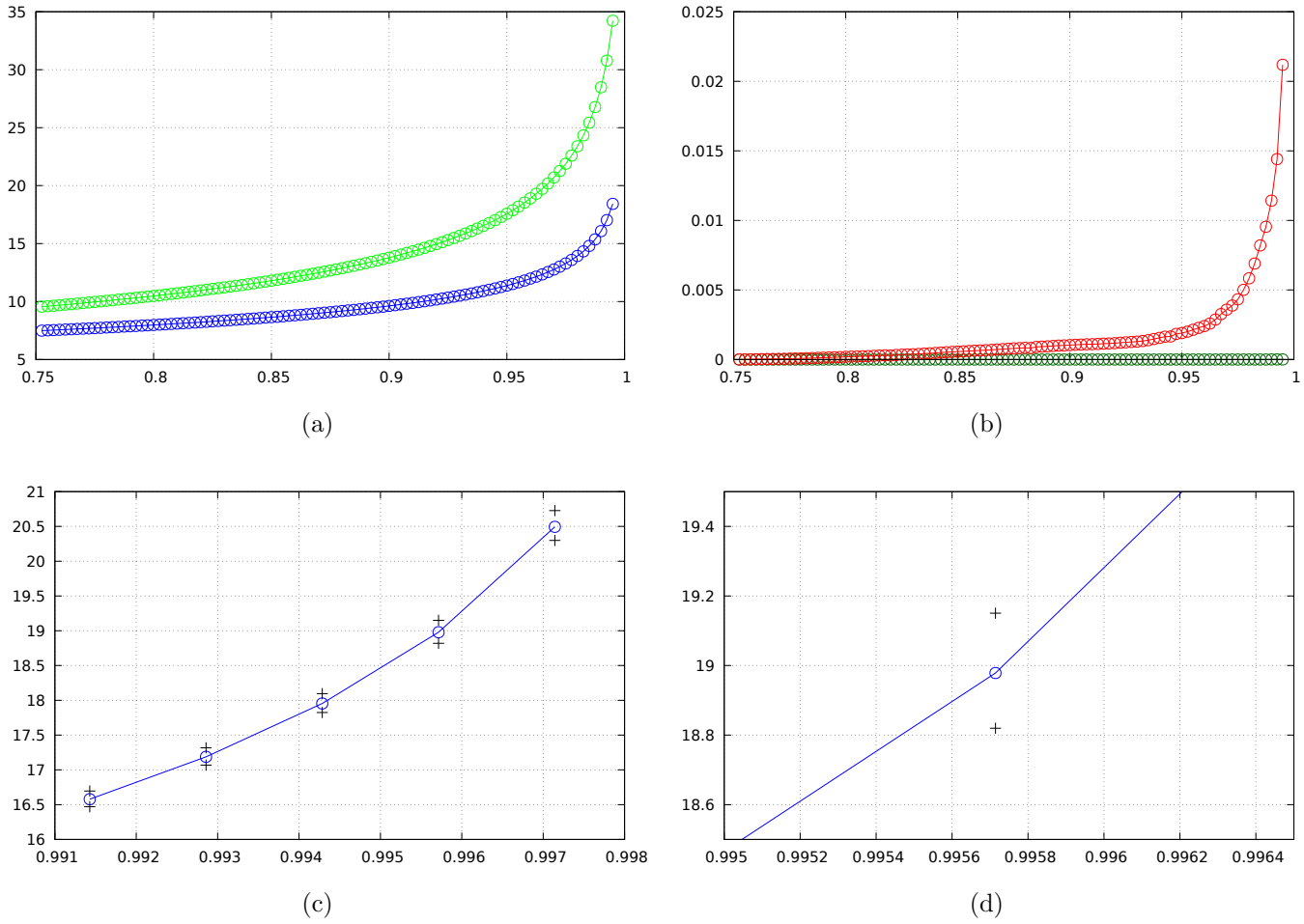



Figure 2: (a) Degree $n = 20$ approximations $\widetilde{\mathrm{CTE}}_q[S_3^c]$ (green) and $\widetilde{\mathrm{CTE}}_q[S_3]$ (blue), $\sigma_1 = \sigma_2 = \sigma_3 = 0.83$. (b) The absolute errors between $\mathrm{CTE}_q[S_3]$ as computed by numerical integration and: (i) $\widetilde{\mathrm{CTE}}_q[S_3]$ (dark green); (ii) the average of 50 MC derived values of $\mathrm{CTE}_q[S_3]$ (red) (c) $\widetilde{\mathrm{CTE}}_q[S_3]$ (blue line) compared to the maximum and minimum of $\mathrm{CTE}_q[S_3]$ computed using 50 MC simulations. (d) A close-up of (c) at one point.

In the next example we generalize Example 1 by considering (i) log-normally distributed risks that are not identically distributed, and (ii) a risk measure that quantifies the variability of the tail risk.

**Example 2.** As before, we start by considering IRM equation (23) with log-normally distributed constituents. Let $e_2(s) := \mathbb{E}[S_N^2 \mathbf{1}\{S_N > s\}],\ s \geq 0$, then - the proof is similar to the one of equation (32) - we readily obtain

$$e_2(s) = \mathbb{E}[S_N^2]\mathcal{L}^{-1}\left\{\frac{1-\phi_{S_N^{*(2)}}(z)}{z}\right\}(s) \text{ for } s \geq 0. \tag{34}$$

Hence, for $q \in [0,\ 1)$, we have

$$\mathbb{E}[S_N^2|\ S_N > \mathrm{VaR}_q[S_N]] \ = \ \frac{\mathbb{E}[S_N^2]}{1-q}\mathcal{L}^{-1}\left\{\frac{1-\phi_{S_N^{*(2)}}(z)}{z}\right\}(\mathrm{VaR}_q[S_N]),$$

which can be simplified further using Theorem 2. Indeed for the special case of interest herein, that is, for $N = 3$, $S_3 = X_{\sigma_1} + X_{\sigma_2} + X_{\sigma_3}$, where $X_{\sigma_1}$, $X_{\sigma_2}$ and $X_{\sigma_3}$ are independent but not necessarily identically log-normally distributed risks, we have

$$\begin{aligned}
\mathbb{E}[S_3^2]\phi_{S_3^{*(2)}}(z) \ &= \ \mathbb{E}[X_{\sigma_1}^2]\phi_{c_1X_{\sigma_1}+X_{\sigma_2}+X_{\sigma_3}}(z) + \mathbb{E}[X_{\sigma_2}^2]\phi_{X_{\sigma_1}+c_2X_{\sigma_2}+X_{\sigma_3}}(z) + \mathbb{E}[X_{\sigma_3}^2]\phi_{X_{\sigma_1}+X_{\sigma_2}+c_3X_{\sigma_3}}(z) \\
&+ \ 2\mathbb{E}[X_{\sigma_1}]\mathbb{E}[X_{\sigma_2}]\phi_{c_1X_{\sigma_1}+c_2X_{\sigma_2}+X_{\sigma_3}}(z) + 2\mathbb{E}[X_{\sigma_1}]\mathbb{E}[X_{\sigma_3}]\phi_{c_1X_{\sigma_1}+X_{\sigma_2}+c_3X_{\sigma_3}}(z) \\
&+ \ 2\mathbb{E}[X_{\sigma_2}]\mathbb{E}[X_{\sigma_3}]\phi_{X_{\sigma_1}+c_2X_{\sigma_2}+c_3X_{\sigma_3}}(z) \text{ for } \mathrm{Re}(z) \geq 0,
\end{aligned}$$

which is formulated in terms of the Laplace transforms of log-normal convolutions, and thus can be approximated with the help of Theorem 1. Bearing this, as well as (33), in mind, we obtain that the approximation for the mTV risk measure is

$$\widetilde{\mathrm{mTV}}_q[S_3] = \frac{1}{\widetilde{\mathrm{CTE}}_q[S_3]} \times \frac{\mathbb{E}[S_3^2]}{1-q}\mathcal{L}^{-1}\left\{\frac{1-\widetilde{\phi}_{S_3^{*(2)}}(z)}{z}\right\}(\widetilde{\mathrm{VaR}}_q[S_3]) \tag{35}$$

for $q \in [0,\ 1)$.

Set $\sigma_1 = 0.81,\ \sigma_2 = 0.83$, and $\sigma_3 = 0.85$ (these choices of parameters are motivated by the empirical findings in O'Neill and Wells (1972)). The approximations $\widetilde{\mathrm{VaR}}_q[S_3]$, $\widetilde{\mathrm{CTE}}_q[S_3]$, and $\widetilde{\mathrm{mTV}}_q[S_3]$ are presented in Figures 3a, 4a, and 4e, respectively. In Figures 3b and 4b we compute the absolute errors between the just-mentioned approximating risk measures and those computed with the help of the MC approach (average of 50 MC simulated values of VaR, CTE, and mTV, each generated with $10^6$ random samples). This completes Example 2.

Given that the aggregate EC $H[S_N]$ has been determined, a somewhat more involved problem is the allocation of this EC to constituents. Namely, it is often of interest to assess the risk contribution of

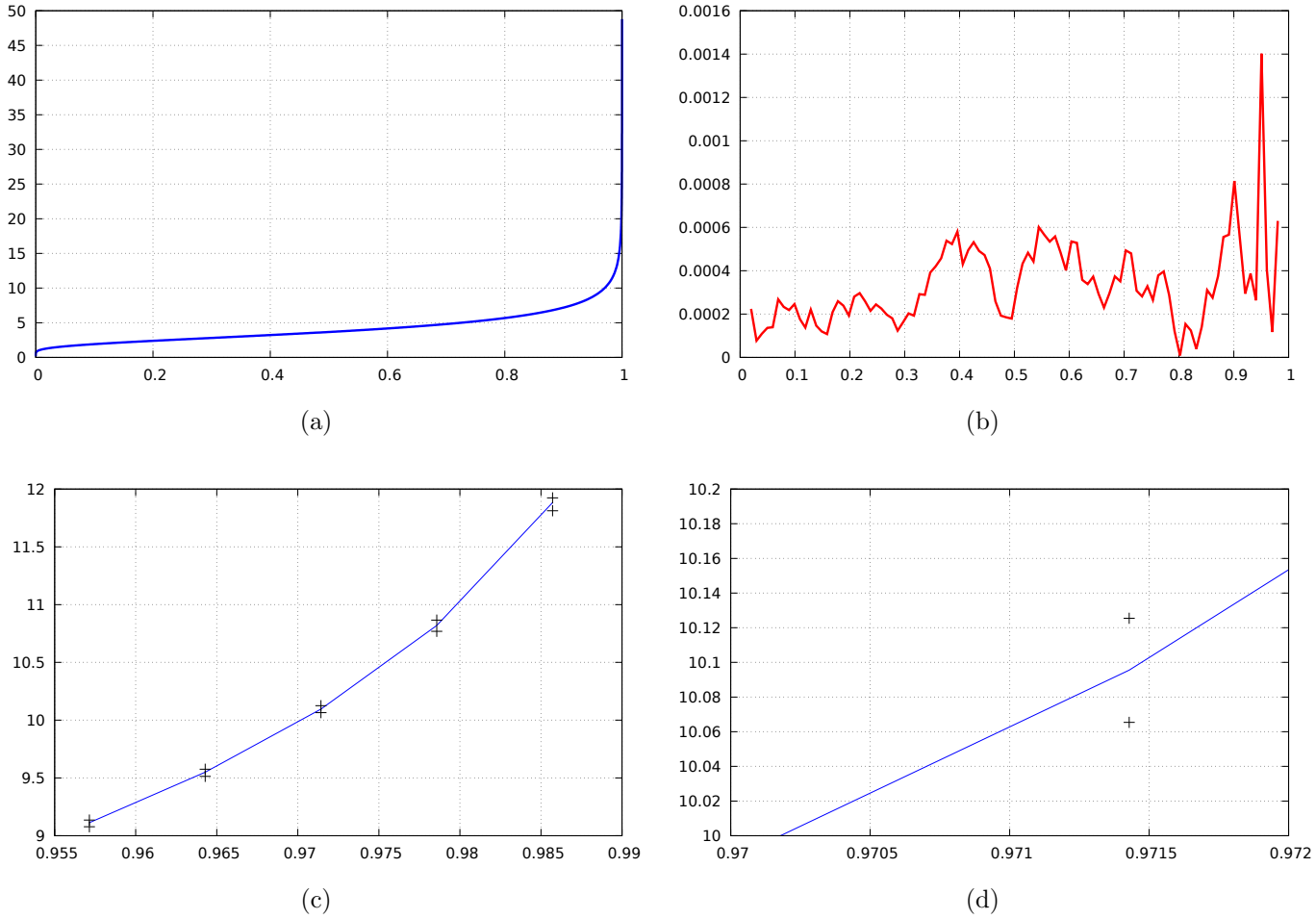
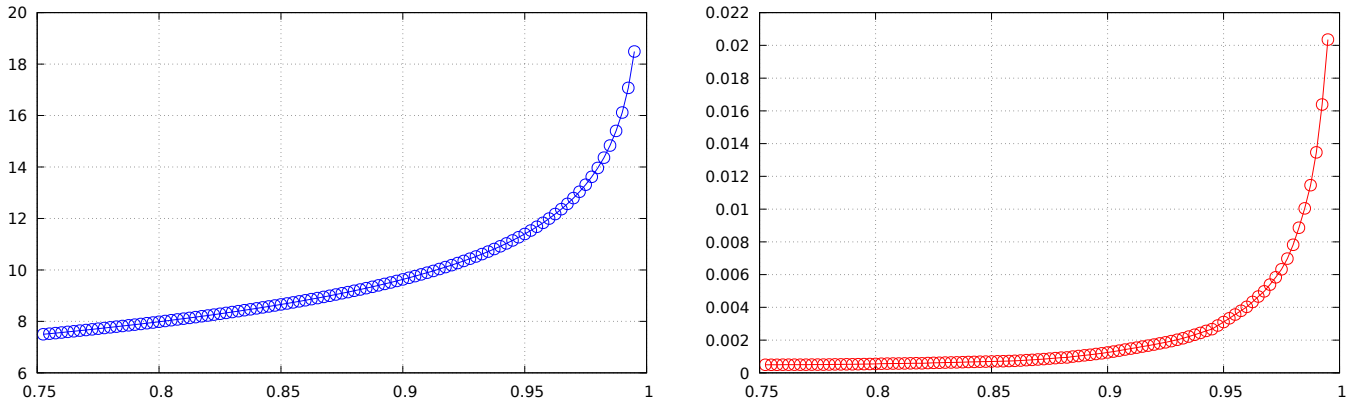


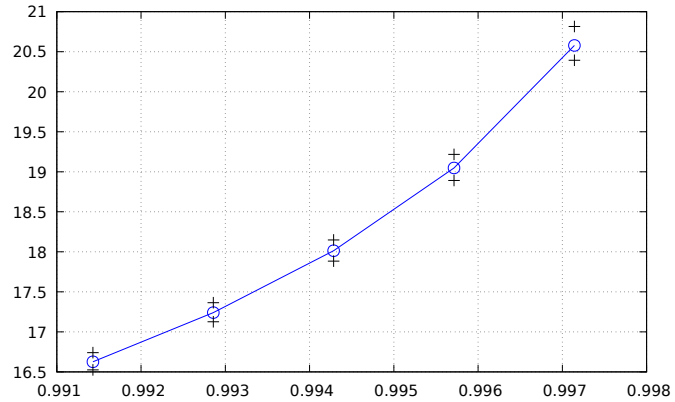

Figure 3: (a) Degree $n = 20$ approximation $\widetilde{\mathrm{VaR}}_q[S_3]$, where $\sigma_1 = 0.81$, $\sigma_2 = 0.83$ and $\sigma_3 = 0.85$. (b) Absolute error between $\widetilde{\mathrm{VaR}}_q[S_3]$ and the average of 50 MC simulations of $\mathrm{VaR}_q[S_3]$. (c) $\widetilde{\mathrm{VaR}}_q[S_3]$ (blue line) compared to the maximum and minimum of $\mathrm{VaR}_q[S_3]$ computed using 50 MC simulations. (d) A close-up of (c) at one point. All simulations are generated using $10^6$ random samples.



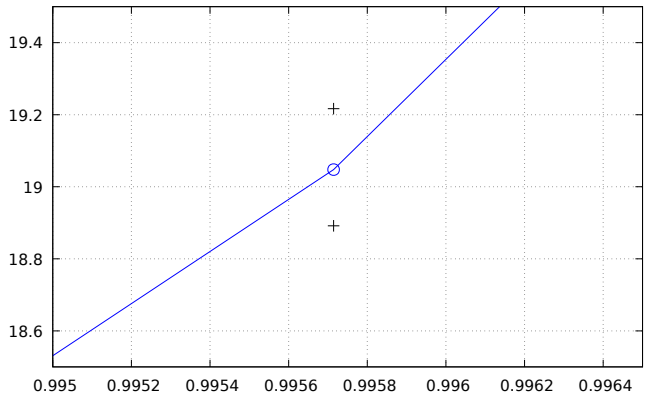

(a)



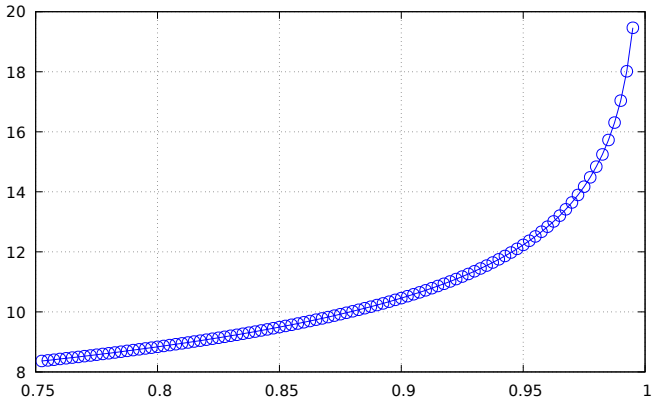

(b)




(c)




(d)




(e)

Figure 4: (a) Degree $n = 20$ approximation $\widetilde{\mathrm{CTE}}_q[S_3]$, where $\sigma_1 = 0.81$, $\sigma_2 = 0.83$ and $\sigma_3 = 0.85$. (b) Absolute error between $\widetilde{\mathrm{CTE}}_q[S_3]$ and the average of 50 MC simulations of $\mathrm{CTE}_q[S_3]$. (c) $\widetilde{\mathrm{CTE}}_q[S_3]$ (blue line) compared to the maximum and minimum of $\mathrm{CTE}_q[S_3]$ computed using 50 MC simulations. (d) A close-up of (c) at one point. (e) Degree $n = 20$ approximation $\widetilde{\mathrm{mTV}}_q[S_3]$ where $\sigma_1 = 0.81$, $\sigma_2 = 0.83$ and $\sigma_3 = 0.85$. All simulations are generated using $10^6$ random samples.

each summand in (23) to $H[S_N]$. More formally, the allocation rule $A : \mathcal{X} \times \mathcal{X} \to [0, \infty]$ such that $A[X, X] = H[X]$ for $X \in \mathcal{X}$ assigns a finite (or infinite) value of the allocated EC to random pairs $(X, S)$ (e.g., Denault, 2001; Furman and Zitikis, 2008). One goal of the allocation exercise is profitability testing, others are cost sharing and pricing (e.g., Venter, 2004).

Clearly, there are infinitely many ways to allocate the aggregate EC, and the literature on the allocation rules is vast and growing rapidly (e.g., Dhaene et al., 2012, and references therein). In this paper we work with the allocation counterparts of risk measures (24) and (25), which are, respectively,

$$\mathrm{CTE}_q[X_i, S_N] = \mathbb{E}[X_i |\ S_N > \mathrm{VaR}_q[S_N]], \tag{36}$$

and

$$\mathrm{mTCoV}_q[X_i, S_N] := \mathbb{E}[X_i |\ S_N > \mathrm{VaR}_q[S_N]] + \frac{\mathbf{Cov}[X_i,\ S_N |\ S_N > \mathrm{VaR}_q[S_N]}{\mathbb{E}[S_N |\ S_N > \mathrm{VaR}_q[S_N]]}, \tag{37}$$

where $i = 1, \ldots, N,\ q \in [0,\ 1)$, and we assume that all the involved quantitites are well-defined and finite. Obviously, we have $\mathrm{CTE}_q[S_N,\ S_N] = \mathrm{CTE}_q[S_N]$ and $\mathrm{mTCoV}_q[S_N, S_N] = \mathrm{mTV}_q[S_N]$ for $i = 1, \ldots, N,\ q \in [0,\ 1)$, and allocation rules (36) and (37) are fully additive. Moreover, these allocation rules are 'weighted' (Furman and Zitikis, 2008), and they are therefore optimal in the sense of Dhaene et al. (2012).

We are now ready to delve into the approximation of allocation rules (36) and (37).

**Example 3.** As in all previous examples, we consider a general IRM with log-normally distributed standalone risks, and we specialize thereafter. Let $h_0(s) := \mathbb{E}[X_j \mathbf{1}\{S_N > s\}],\ s \geq 0$, then - the proof is at the end of this section,

$$h_0(s) = \mathbb{E}[X_j] \mathcal{L}^{-1} \left\{ \frac{1}{z} \left( 1 - \phi_{S_N - X_j + X_j^*}(z) \right) \right\} (s) \text{ for } s \geq 0. \tag{38}$$

This implies immediately, for $q \in [0,\ 1)$ and $c_j = \exp(\sigma_j^2)$,

$$\mathrm{CTE}_q[X_j,\ S_N] = \frac{\mathbb{E}[X_j]}{1-q} \mathcal{L}^{-1} \left\{ \frac{1}{z} \left( 1 - \phi_{\sum_{i=1,\ i \neq j}^{N} X_i + c_j X_j}(z) \right) \right\} (\mathrm{VaR}_q[S_N]).$$

Further, set $N = 3$, then

$$\mathrm{CTE}_q[X_j,\ S_3] \approx \frac{\mathbb{E}[X_j]}{1-q} \mathcal{L}^{-1} \left\{ \frac{1}{z} \left( 1 - \widetilde{\phi}_{\sum_{i=1,\ i \neq j}^{3} X_i + c_j X_j}(z) \right) \right\} (\widetilde{\mathrm{VaR}}_q[S_3]) = \widetilde{\mathrm{CTE}}_q[X_j,\ S_3],, \tag{39}$$

which is computed evoking Theorem 1.

In order to approximate the modified Tail Covariance allocation rule, we recall the following trivial equation

$$\begin{aligned} \mathbf{Cov}[X_j,\ S_N |\ S_N > \mathrm{VaR}_q[S_N]] &= \mathbb{E}[X_j S_N |\ S_N > \mathrm{VaR}_q[S_N]] \\ &- \mathbb{E}[X_j |\ S_N > \mathrm{VaR}_q[S_N]] \times \mathbb{E}[S_N |\ S_N > \mathrm{VaR}_q[S_N]] \text{ for } q \in [0,\ 1). \end{aligned}$$

The product of expectations can be computed as in Example 1, hence, we only need to approximate the mixed expectation. To this end, let $h(s) := \mathbb{E}[XS\mathbf{1}\{S > s\}],\ s \geq 0$, and note - the proof is at the end of this section,

$$\begin{aligned} h(s) &= \mathbb{E}[X_j]\mathbb{E}[S_{N,-j}]\mathcal{L}^{-1}\left\{\frac{1}{z}\left(1-\phi_{S^*_{N,-j}+X^*_j}(z)\right)\right\}(s) \\ &\quad + \mathbb{E}[X_j^2]\mathcal{L}^{-1}\left\{\frac{1}{z}\left(1-\phi_{S_{N,-j}+X_j^{*(2)}}(z)\right)\right\}(s), \end{aligned}$$

where $S_{N,-j} := \sum_{i=1,\ i\neq j}^{N} X_i$. This can be simplified with the help of Theorem 2, and in particular, for $N = 3$, $c_j$ as before, and $s \geq 0$,

$$\begin{aligned} h(s) &= \mathbb{E}[X_j]\mathbb{E}[S_{3,-j}]\mathcal{L}^{-1}\left\{\frac{1}{z}\left(1-\sum_{i=1,\ i\neq j}^{3}\frac{\mathbb{E}[X_i]}{\mathbb{E}[S_{3,-j}]}\phi_{c_jX_{\sigma_j}+c_iX_{\sigma_i}+\sum_{k=1,\ k\neq i,j}^{3}X_{\sigma_k}}(z)\right)\right\}(s) \\ &\quad + \mathbb{E}[X_j^2]\mathcal{L}^{-1}\left\{\frac{1}{z}\left(1-\phi_{S_{3,-j}+c_j^2X_j}(z)\right)\right\}(s), \end{aligned}$$

which implies, for $c_j = \exp(\sigma_j^2)$ and $q \in [0,\ 1)$,

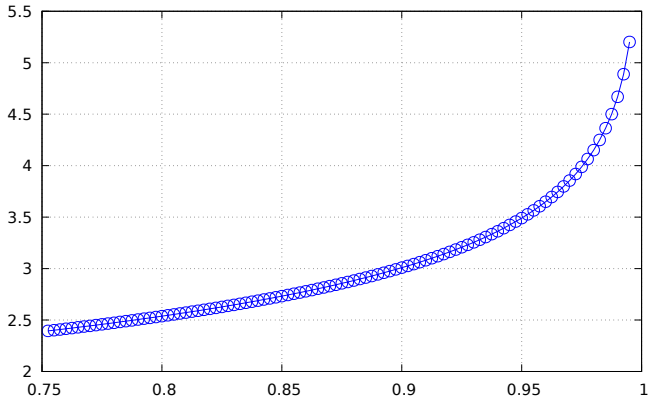
$$\begin{aligned} &\mathbb{E}[X_jS_3|\ S_3 > \mathrm{VaR}_q[S_3]] \qquad (40)\\ &= \frac{\mathbb{E}[X_j]\mathbb{E}[S_{3,-j}]}{1-q}\mathcal{L}^{-1}\left\{\frac{1}{z}\left(1-\sum_{i=1,\ i\neq j}^{3}\frac{\mathbb{E}[X_i]}{\mathbb{E}[S_{3,-j}]}\phi_{c_jX_{\sigma_j}+c_iX_{\sigma_i}+\sum_{k=1,\ k\neq i,j}^{3}X_{\sigma_k}}(z)\right)\right\}(\mathrm{VaR}_q[S_3]) \\ &\quad + \frac{\mathbb{E}[X_j^2]}{1-q}\mathcal{L}^{-1}\left\{\frac{1}{z}\left(1-\phi_{S_{3,-j}+c_j^2X_j}(z)\right)\right\}(\mathrm{VaR}_q[S_3]) \\ &\approx \frac{\mathbb{E}[X_j]\mathbb{E}[S_{3,-j}]}{1-q}\mathcal{L}^{-1}\left\{\frac{1}{z}\left(1-\sum_{i=1,\ i\neq j}^{3}\frac{\mathbb{E}[X_i]}{\mathbb{E}[S_{3,-j}]}\widetilde{\phi}_{c_jX_{\sigma_j}+c_iX_{\sigma_i}+\sum_{k=1,\ k\neq i,j}^{3}X_{\sigma_k}}(z)\right)\right\}(\mathrm{VaR}_q[S_3]) \\ &\quad + \frac{\mathbb{E}[X_j^2]}{1-q}\mathcal{L}^{-1}\left\{\frac{1}{z}\left(1-\widetilde{\phi}_{S_{3,-j}+c_j^2X_j}(z)\right)\right\}(\widetilde{\mathrm{VaR}}_q[S_3]), \end{aligned}$$

which establishes the approximation of the conditional covariance, and can be computed using Theorem 1. Finally, the desired approximation of the modified Tail Covariance is

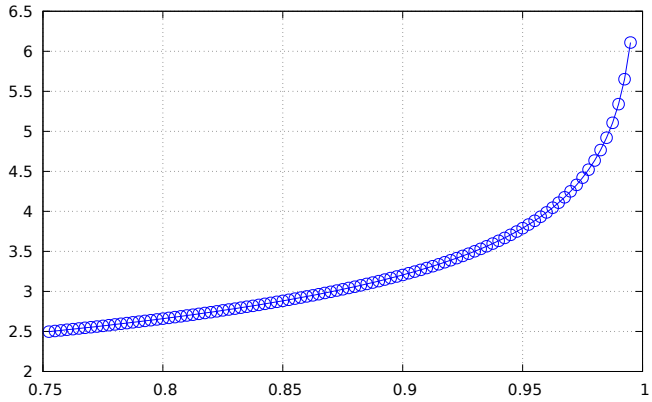
$$\widetilde{\mathrm{mTCov}}_q[X_{\sigma_i},\ S_3] = \widetilde{\mathrm{CTE}}_q[X_{\sigma_i}, \tilde{S}_3] + \frac{\widetilde{\mathrm{Cov}}[X_j,\ S_3|\ S_3 > \mathrm{VaR}_q[S_3]]}{\widetilde{\mathrm{CTE}}_q[\tilde{S}_3]}, \qquad (41)$$

where $\widetilde{\mathrm{CTE}}_q[X_{\sigma_i}, \tilde{S}_3]$ and $\widetilde{\mathrm{CTE}}_q[\tilde{S}_3]$ are given in (39) and (33), respectively.

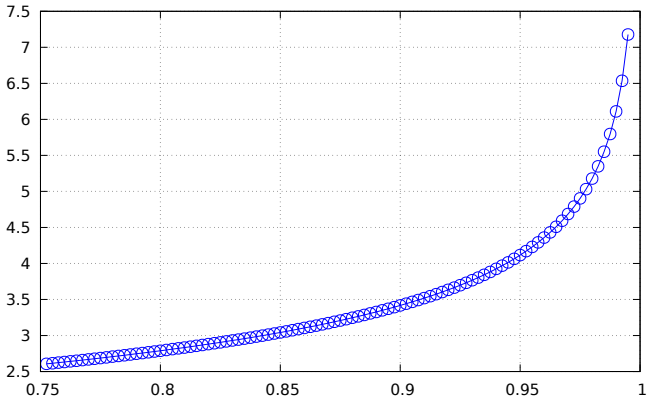
For visualization purposes, we again set $\sigma_1 = 0.81,\ \sigma_2 = 0.83$ and $\sigma_3 = 0.85$, then the approximating allocation rules based on the CTE and the modified Tail Covariance risk measures are depicted in Figures 5 and 6. This completes Example 3.




(a)

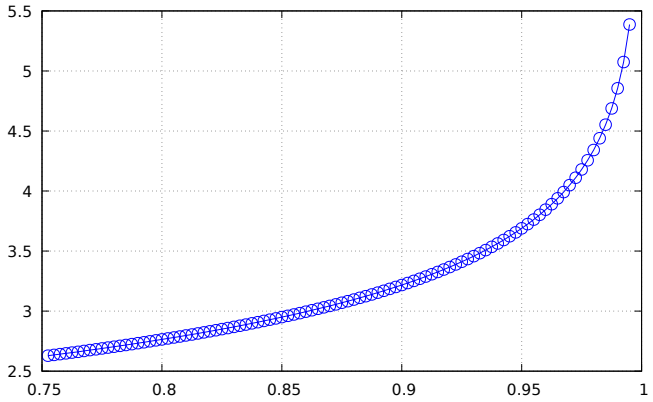



(b)

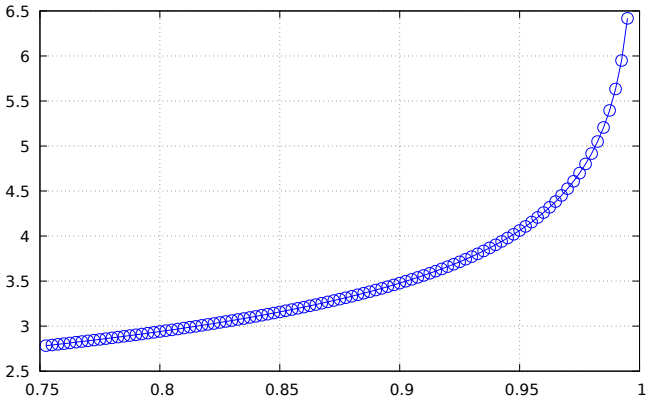



(c)

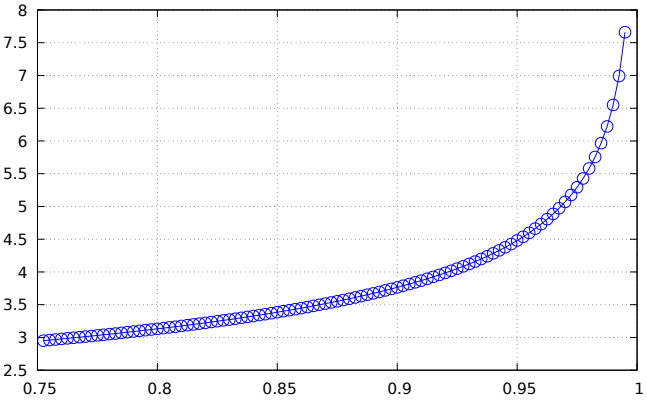
Figure 5: Degree $n = 20$ approximations (a) $\widetilde{\mathrm{CTE}}_q[X_{0.81},\ S_3]$ (b) $\widetilde{\mathrm{CTE}}_q[X_{0.83},\ S_3]$ and (c) $\widetilde{\mathrm{CTE}}_q[X_{0.85},\ S_3]$.




(a)




(b)




(c)

Figure 6: Degree $n = 20$ approximations (a) $\widetilde{\mathrm{mTCov}}_q[X_{0.81},\ S_3]$ (b) $\widetilde{\mathrm{mTCov}}_q[X_{0.83},\ S_3]$ (c) $\widetilde{\mathrm{mTCov}}_q[X_{0.85},\ S_3]$.

*Proof of Equation (32).* Note that the Laplace transform of $e(s)$ is

$$\begin{aligned}\mathcal{L}\{e(s)\}(z) &= \int_0^\infty e^{-zs}e(s)ds = \int_0^\infty e^{-zs}\left(\int_s^\infty tf(t)dt\right)ds = \frac{1}{z}\int_0^\infty (1-e^{-zt})tf(t)dt \\ &= \frac{\mathbb{E}[S_N]}{z}\int_0^\infty (1-e^{-zt})\mathrm{d}F_{S_N^*}(t) = \mathbb{E}[S_N]\left(\frac{1-\phi_{S_N^*}(z)}{z}\right)\end{aligned}$$

for $\mathrm{Re}(z) \geq 0$. This completes the proof of the equation. □

*Proof of Equation* (38). We have, for $j = 1, \ldots, N$,

$$\begin{aligned}\mathcal{L}\{h_0(s)\}(z) &= \int_0^\infty e^{-zs}h_0(s)\mathrm{d}s = \int_0^\infty e^{-zs}\int_0^\infty\int_s^\infty uf_{X_j,S_N}(u,v)\mathrm{d}v\mathrm{d}u\mathrm{d}s \\ &= \int_0^\infty e^{-zs}\int_0^\infty uf_{X_j}(u)\left(\int_{s-u}^\infty f_{S_N-X_j}(v)\mathrm{d}v\right)\mathrm{d}u\mathrm{d}s \\ &= \mathbb{E}[X_j]\int_0^\infty e^{-zs}\int_0^s f_{X_j^*}(u)\overline{F}_{S_N-X_j}(s-u)\mathrm{d}u\mathrm{d}s \\ &= \mathbb{E}[X_j]\int_0^\infty e^{-zs}\overline{F}_{S_N-X_j+X_j^*}(s)\mathrm{d}s\end{aligned}$$

for $\mathrm{Re}(z) \geq 0$. We then use property (ii) in Appendix A, which concludes the proof of the equation. □

*Proof of Equation 40.* We have the following string of equations, for $j = 1, \ldots, N$,

$$\begin{aligned}\mathcal{L}\{h(s)\}(z) &= \int_0^\infty e^{-zs}h(s)\mathrm{d}s = \int_0^\infty e^{-zs}\mathbb{E}[X_jS_N\mathbf{1}\{S>s\}]\mathrm{d}s \\ &= \int_0^\infty e^{-zs}\int_0^\infty\int_s^\infty uvf_{X_j,S_N}(u,v)\mathrm{d}v\mathrm{d}u\mathrm{d}s \\ &= \int_0^\infty e^{-zs}\int_0^\infty uf_{X_j}(u)\left(\int_s^\infty vf_{S_N-X_j}(v-u)\mathrm{d}v\right)\mathrm{d}u\mathrm{d}s \\ &= \mathbb{E}[X_j]\mathbb{E}[S_{N,-j}]\int_0^\infty e^{-zs}\int_0^s f_{X_j^*}(u)\overline{F}_{S_{N,-j}^*}(s-u)\mathrm{d}u\mathrm{d}s \\ &+ \mathbb{E}[X_j^2]\int_0^\infty e^{-zs}\int_0^s f_{X_j^{*(2)}}(u)\overline{F}_{S_{N,-j}}(s-u)\mathrm{d}u\mathrm{d}s \\ &= \mathbb{E}[X_j]\mathbb{E}[S_{N,-j}]\int_0^\infty e^{-zs}\overline{F}_{S_{N,-j}^*+X_j^*}(s)\mathrm{d}s + \mathbb{E}[X_j^2]\int_0^\infty e^{-zs}\overline{F}_{S_{N,-j}+X_j^{*(2)}}(s)\mathrm{d}s\end{aligned}$$

for $\mathrm{Re}(z) \geq 0$. This, along with property (ii) in Appendix A, completes the proof of the equation. □

# 6 Aggregate economic capital determination: collective risk model

In this section we assume that $N$ is a discrete r.v., and so $S_N = X_{\sigma_1} + \cdots + X_{\sigma_N}$ is a collective risk model with all of $X_{\sigma_1},\ X_{\sigma_2}, \ldots$ independent mutually and on $N$, as well as identically distributed as a canonical r.v. $X_\sigma \sim LN(\sigma^2),\ \sigma > 0$. Let $G_N(z) := \mathbb{E}[z^N]$ denote the probability generating function (p.g.f.) of $N$ for all $z \in (-\infty,\ \infty)$ such that the p.g.f. is well-defined and finite.

In principle, once we have fixed the distributions of $X_\sigma$ and $N$, we may proceed exactly as before and approximate the Laplace transform of $S_N$ with the help of Theorem 1. This is clear from

$$\phi_{S_N}(z) = G_N(\phi_{X_\sigma}(z)) \approx G_N(\widetilde{\phi}_{X_\sigma}(z)) = \widetilde{\phi}_{S_N}(z),\ \mathrm{Re}(z) \geq 0.$$

However, we must make a slight adjustment when computing the c.d.f $F_{S_N}$ since the measure $\mu(\mathrm{d}x) := \mathbb{P}(S_N \in \mathrm{d}x)$ may have an atom at zero, and consequently $F_{S_N}$ may be discontinuous there. We find that better numerical results are achieved if we remove this atom.

More specifically, let $\mu(\{0\}) = p_0 \in (0,\ 1)$. The measure $\mu_0(\mathrm{d}x) := \mu(\mathrm{d}x) - p_0\delta_0(\mathrm{d}x)$ is absolutely continuous with respect to the Lebesgue measure. It is easy to see that $F_0(x) := \mu_0([0, x])$ has Laplace transform $\phi_0(z) := (\phi_{S_N}(z) - p_0)/z,\ \mathrm{Re}(z) \geq 0$. Therefore, we can use

$$F_{S_N}(x) = p_0 + \mathcal{L}^{-1}\left\{\phi_0(z)\right\}(x)$$

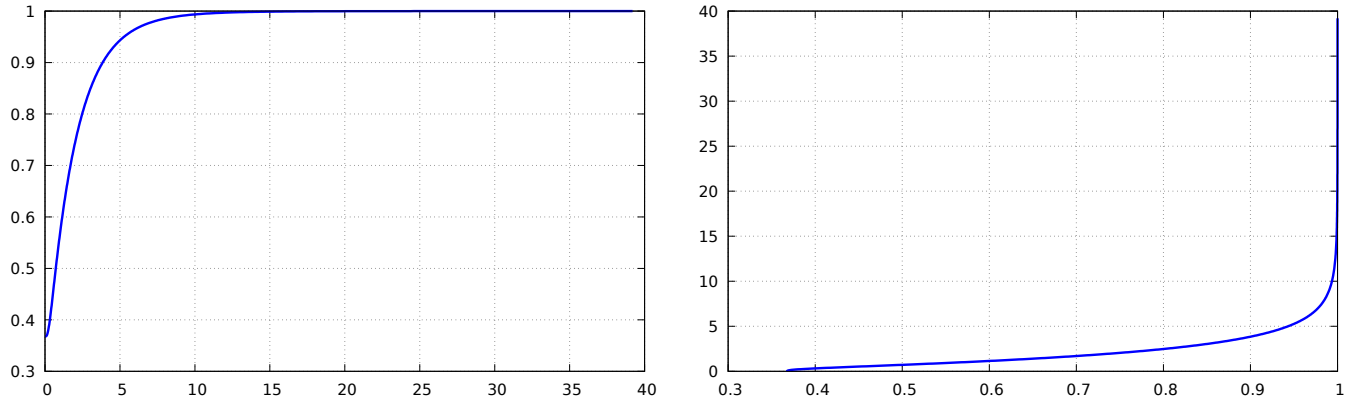
for the values of $x$ near zero, and we revert to the methods in our preliminary examples otherwise.

**Example 4.** Let $N$ be distributed Poisson with the unit rate parameter, and let $S_N = X_{\sigma_1} + X_{\sigma_2} + \cdots$, where $X_{\sigma_1},\ X_{\sigma_2}, \ldots$ are mutually independent, independent of $N$, and such that $\sigma_1 = \sigma_2 = \cdots = 0.83$. For this special $N$ r.v., we have $p_0 = \exp(-1)$. The graph of the approximation $\widetilde{F}_{S_N}(x)$ is shown in Figure 7. Also shown therein is $\widetilde{\mathrm{VaR}}_q[S_N]$, which can be computed as the inverse of $\widetilde{F}_{S_N}(x)$ as before. Of course in this case we have $\mathrm{VaR_q[S_N]} = 0$ for $q \leq \exp(-1)$. Using these results, we also compute $\widetilde{\mathrm{CTE}}_q[S_N]$ and $\widetilde{\mathrm{mTV}}_q[\tilde{S}_N]$, which are depicted in Figure 8. This completes Example 4.

# 7 Computation time

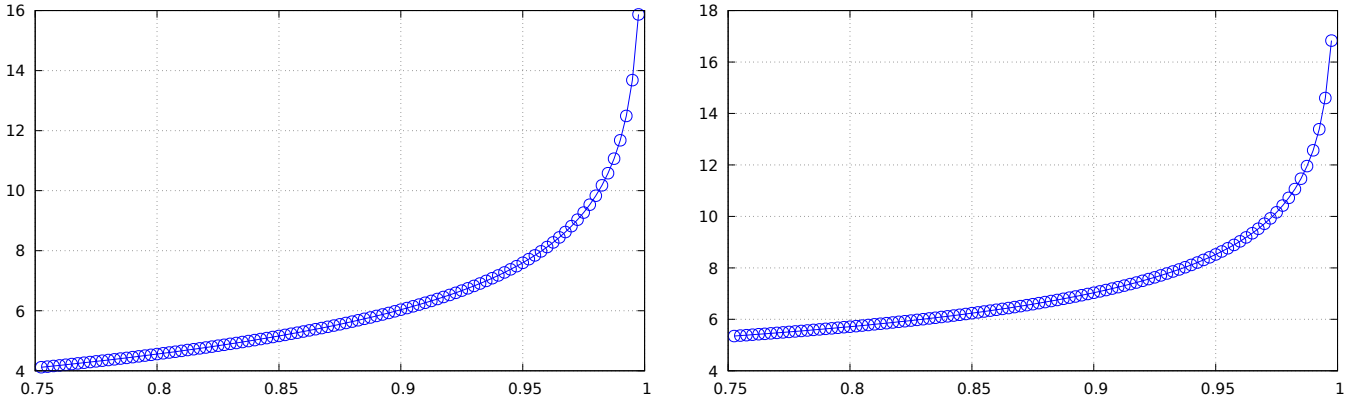
In this section we provide the computation time needed to compute the quantities of interest in the previous sections. These computation times are presented in Table 1.

All calculations were carried out on a desktop computer with 32GB of memory and an Intel i7-2600K 3.40GHz CPU. Times shown are measured in seconds. These represent the number of seconds required




(a) (b)

Figure 7: Degree $n = 20$ approximations (a) $\widetilde{F}_{S_N}(x)$ and (b) $\widetilde{\mathrm{VaR}}_q[S_N]$.




(a) (b)

Figure 8: Degree $n = 20$ approximations (a) $\widetilde{\mathrm{CTE}}_q[S_N]$ and (b) $\widetilde{\mathrm{mTV}}_q[S_N]$.

| Quantity | Pre-Comp. Integ. | W/O Pre-Comp. Integ. | $10^6$ MC | $10^7$ MC |
|---|---|---|---|---|
| $\widetilde{\mathrm{VaR}}_{0.5}[S_3]$ | 0 | N/A | 0 | 0 |
| $\widetilde{\mathrm{CTE}}_{0.5}[S_3]$ | 0.56 | 7.41 | 0.35 | 3.46 |
| $\widetilde{\mathrm{mTV}}_{0.5}[S_3]$ | 1.06 | 211.83 | 0.36 | 3.46 |
| $\widetilde{\mathrm{CTE}}_{0.5}[X_{0.81},\, S_3]$ | 0.92 | 6.38 | 0.36 | 3.53 |
| $\widetilde{\mathrm{mTCov}}_{0.5}[X_{0.81},\, S_3]$ | 3.25 | 43.67 | 0.35 | 3.45 |

Table 1: Computation times for various quantities of interest in the paper.

to compute one value of the quantity listed in the left-most column by the method listed in the top-most row. None of the times include the time required to compute the coefficients of the approximation – these were pre-computed and stored on hard drive (the coefficients are available for download at www.math.yorku.ca/~akuznets/math.html for a range of values of $\sigma \in (0, 3)$ and for $n = 10$ and $n = 20$). The column "Pre-Comp. Integ." shows the timings for when we have pre-computed and stored the integrands for the various Laplace inversions we have to do. Pre-computing these integrands takes a rather long time (due to the fact that we use a large number of discretization points in the inverse Laplace transform – see Appendix A), and if we compute multiple quantities of interest - and this is the case herein - it is more efficient to pre-compute these integrands and store them rather then compute them every time. For a comparison, the column "W/O Pre-Comp. Inteq." shows the computation time that includes the time used to build the integrands.

The two columns on the right show the time taken by the Monte Carlo method. We emphasize that if we do not take into account the pre-computation time of the integrands, our method is faster than the Monte Carlo method and is much more accurate (e.g., Figure 2).

# 8 Concluding discussion

Convolutions of log-normally distributed r.v.'s play a prominent role in actuarial science, and well-beyond it. Nevertheless, the existing methods, which as a rule hinge on either one of (i) the moment matching technique, (ii) series expansion of the Laplace transform, and (iii) asymptotic analysis, may deliver inaccurate results. Rather unfortunately, the just-mentioned inaccuracies exacerbate when, e.g., the tail risk is of interest, and this is precisely the phenomenon that concerns the modern insurance

regulation the most. In this paper we have proposed a hybrid approach to resolving the problem. More specifically, we have shown that it is possible to approximate the distribution of the sum of independent and log-normally distributed r.v.'s with the help of the distribution of $n$-fold gamma convolutions in such a way that the two are arbitrarily close. We then have utilized the class of Padé approximations to find the parameters of the approximating distribution. For the convenience of the end-user, we have pre-computed these parameters for a range of values of $\sigma \in (0, 3)$ and for $n = 10$ and $n = 20$: these can be downloaded at `www.math.yorku.ca/~akuznets/math.html`.

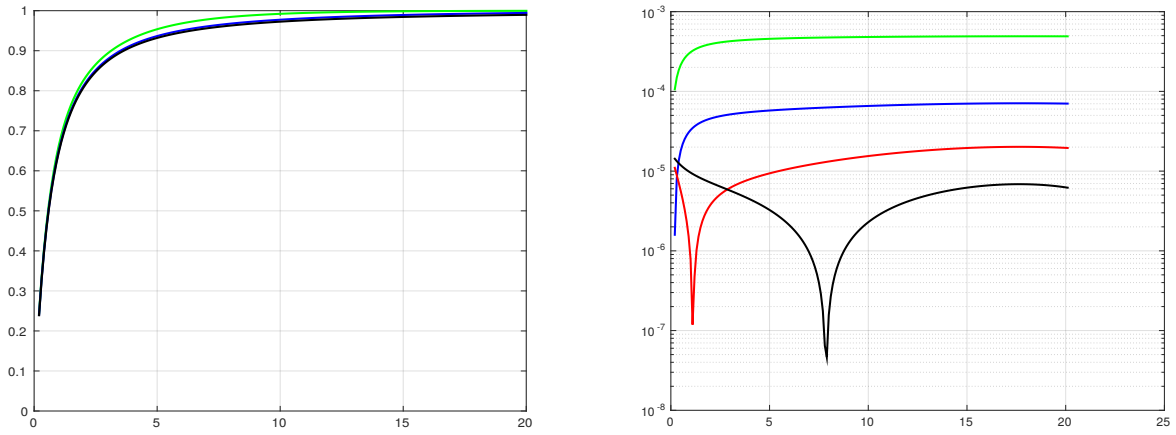
Remarkably, the algorithm that arises from our method is fast, accurate, and, last but not least, very versatile. We have discussed the two former advantages earlier in the paper, and so we touch on the versatility in more detail here. In this respect, recall that the thrust of our method is the observation that the log-normal distribution is a member of the encompassing class of the generalized gamma convolutions. Indeed this is how the link to the finite gamma convolutions emerges. Another distribution that belongs to the class of generalized gamma convolutions - and is popular in actuarial science - is Pareto (e.g., Bondesson, 1992).

Consider the following p.d.f. of a Pareto distributed r.v. $Y$ with shape parameter $\alpha > 0$

$$f(y) = \alpha(1 + y)^{-\alpha-1}, \quad y > 0. \tag{42}$$

Pareto is a classical example of a heavy-tailed distribution, and it holds that for the values $\alpha \in (1,\ 2]$ the variance of $Y$ is infinite, and for the values $\alpha \in (0,\ 1]$ even the mean of $Y$ is infinite. We now quickly apply our method to illustrate how it can be used in order to approximate p.d.f. (42), as well as the corresponding c.d.f. We note in passing that the approximating c.d.f. of the sum of independent but not necesserily identically Pareto-distributed r.v.'s can be computed along the line of Corollary 1 (see, e.g., Ramsay, 2009, that considers the situation of i.i.d. summands). In Seal (1980), it is mentioned that in the real-world applications (e.g., fire and auto-mobile losses, to name a few), insurers have to frequently deal with Pareto-distributed risks with infinite variances and finite means. Therefore in our illustration below we choose $\alpha = 1.5$.

We set $z^* = 1$, run the approximation algorithm from Section 3 for $n = \{2,\ 4,\ 6,\ 8,\ 10,\ 20,\ 30,\ 40\}$ in order to find the approximating Laplace transform, and then use the inversion method from Appendix A to find the p.d.f. We compute the approximating c.d.f. over 200 points equally spaced over the interval $[0,\ 20]$, and compare the result with the exact one that follows from formula (42). The outcomes are depicted in Figure 9. Noticeably, even lower orders of approximations, provide practically visually indistinguishable results.



(a) (b)

Figure 9: (a) The black curve is the exact Pareto c.d.f. (with $\alpha = 1.5$), the green (respectively, blue) curve is the c.d.f. of the approximations $Y_2$ (respectively, $Y_4$). (b) The difference between the Pareto c.d.f. and the c.d.f. of $Y_n$ for $n \in \{10, 20, 30, 40\}$ (larger values of $n$ correspond to smaller errors).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# Appendix A Numerical inversion of Laplace transform

In this section we remind the reader some results from the theory of Laplace transform and discuss how to compute the inverse Laplace transform efficiently and accurately. Let $f$ be a p.d.f. and define by $\phi$ the corresponding Laplace transform

$$\phi(z) = \mathcal{L}f(z) = \int_0^\infty e^{-zx} f(x)\mathrm{d}x, \quad \mathrm{Re}(z) \geq 0.$$

The following results are well-known:

(i) If $f$ is sufficiently smooth, then

$$f(x) = \mathcal{L}^{-1}\{\phi(z)\}(x) = \frac{1}{2\pi i}\int_{c-\mathrm{i}\infty}^{c+\mathrm{i}\infty} e^{zx}\phi(z)\mathrm{d}z, \tag{43}$$

where $c$ is an arbitrary positive number.

(ii) Let $F$ be the c.d.f. corresponding to $f$ and $\bar{F} := 1 - F$. Then $\mathcal{L}F(z) = \phi(z)/z$ and $\mathcal{L}\bar{F}(z) = (1-\phi(z))/z$.

Let us now discuss how to compute the inverse Laplace transform efficiently. Assume that the function $\phi$ is analytic in $\mathbb{C}\setminus(-\infty, 0]$ and it converges to zero uniformly as $z \to \infty$ in this domain. Note that these conditions, while rather restrictive in general, are always satisfied in examples used in the present paper (see formula (8), for example). We begin by writing the inverse Laplace transform in an equivalent form

$$f(x) = \mathrm{Im}\left(\frac{1}{\pi}\int_c^{c+\mathrm{i}\infty} e^{zx}\phi(z)\mathrm{d}z\right), \quad x \geq 0, \tag{44}$$

which follows from the fact $\overline{\phi(z)} = \phi(\overline{z})$. Now we choose $a$ in the second quadrant (that is, $\arg(a) \in (\pi/2, \pi)$), rotate the contour of integration and change the variable of integration $z = c + au$ and obtain the following integral representation

$$f(x) = \mathrm{Im}\left(\frac{ae^{cx}}{\pi}\int_0^\infty e^{u\,\mathrm{Re}(a)x} e^{\mathrm{i}u\,\mathrm{Im}(a)x}\phi(c+au)\mathrm{d}u\right), \quad x \geq 0. \tag{45}$$

The above integral is more convenient to work with, compared to (44), for the following reason: the integrand decays exponentially fast in (45) as $\mathrm{Re}(a) < 0$ .

In the computations in Sections 4, 5 and 6, we typically choose $-0.5 \leq \mathrm{Re}(a) \leq -0.1$, $\mathrm{Im}(a) = 1$ and $0.25 \leq c \leq 5$. To compute the oscillatory integral (45), we use the Filon's quadrature (Filon, 1928; Fosdick, 1968). This entails approximating $\phi(c+au)$ locally by a polynomial of degree two (using three discretization points), and integrating the result against the exponential function. In total, we typically truncate the integral in (45) at a point in the interval $[250, 500]$ and use between 500,001 and 1,000,001 unevenly spaced discretization points to evaluate the resulting integral.