Ch 4.3 Bialgebra and double cosets

Shu Xiao Li
York University

June 12, 2015
Recall

- \( G_* = G_0 < G_1 < G_2 < \ldots \) is tower of group where \( G_n \) could be \( S_n \), \( S_n[\Gamma] \) or \( GL_n(\mathbb{F}_q) \).
- \( A = \bigoplus_{n \geq 0} R(G_n) \) where \( A_n = R(G_n) \) is the \( \mathbb{Z} \)-span of \( \text{Irr}(G_n) \).
- Define product and coproduct on \( A \) as
  - \( m = \text{ind}_{i,j}^{i+j} : A_i \otimes A_j \rightarrow A_{i+j} \),
  - \( \Delta = \bigoplus_{i+j=n} \text{res}_{i,j}^{i+j} : A_n \rightarrow \bigoplus_{i+j=n} A_i \otimes A_j \).
- To show \( A \) has bialgebra structure, we need
  - \( \Delta \circ m = (m \otimes m) \circ (\text{id} \otimes T \otimes \text{id}) \circ (\Delta \otimes \Delta) \).
Let $G$ be a group, and $H, K$ its subgroups, then double cosets $H \backslash G / K$ are of the form $HgK = \{hgk \mid h \in H, k \in K\}$ for $g \in G$. Double cosets $H \backslash G / K$ partitions $G$.

$\alpha = (\alpha_1, \ldots, \alpha_k) \in \mathbb{N}^k$ is a composition if $\alpha_i > 0$,

$\tilde{\alpha} = (\tilde{\alpha}_1, \ldots, \tilde{\alpha}_k) \in \mathbb{N}^k$ is a almost-composition if $\tilde{\alpha}_i \geq 0$. 
For almost-partitions $\tilde{\alpha}, \tilde{\beta}$ with $|\tilde{\alpha}| = |\tilde{\beta}| = n$, and matrix $A$ with row sum $\tilde{\alpha}$, column sum $\tilde{\beta}$, we can obtain a permutation $\omega_{A} \in S_{n}$ as follows:

E.g. Let $n = 9$, $\tilde{\alpha} = (4, 5)$, $\tilde{\beta} = (3, 4, 2)$ and $A = \begin{bmatrix} 2 & 2 & 0 \\ 1 & 2 & 2 \end{bmatrix}$, we can get two matrices by filling in numbers $\{1, 2, \ldots, 9\}$ vertically and horizontally i.e.

\[
I_{A} = \begin{bmatrix} \{1, 2\} & \{4, 5\} & \emptyset \\ \{3\} & \{6, 7\} & \{8, 9\} \end{bmatrix} \quad \text{and} \quad J_{A} = \begin{bmatrix} \{1, 2\} & \{3, 4\} & \emptyset \\ \{5\} & \{6, 7\} & \{8, 9\} \end{bmatrix}.
\]

Then, $\omega_{A}$ is the increasing bijection from $I_{A}$ to $J_{A}$ i.e.

\[
\omega_{A} = \begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 & 9 \\ 1 & 2 & 5 & 3 & 4 & 6 & 7 & 8 & 9 \end{pmatrix}.
\]

This process can be reversed under certain conditions.
Proposition 4.23

For almost-partitions $\tilde{\alpha}, \tilde{\beta}$ with $|\tilde{\alpha}| = |\tilde{\beta}| = n$, the permutations $\{\omega_A\}$ as $A$ runs over all matrices satisfying the condition above, give a system of double coset representatives for $S_{\tilde{\alpha}} \backslash S_n / S_{\tilde{\beta}}$, $S_{\tilde{\alpha}}[\Gamma] \backslash S_n[\Gamma] / S_{\tilde{\beta}}[\Gamma]$ and $P_{\tilde{\alpha}} \backslash GL_n / P_{\tilde{\beta}}$. 
For each double coset $S\tilde{\alpha}\omega S\tilde{\beta}$, we construct $\omega_A$ by reordering each part $\tilde{\alpha}_i, \tilde{\beta}_j$ of $\omega$ into increasing order. E.g.

$$
\omega = \begin{pmatrix}
1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 & 9 \\
4 & 8 & 2 & 5 & 3 & 9 & 1 & 7 & 6 \\
\end{pmatrix} \in S_n,
$$

$$
\omega' = \begin{pmatrix}
1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 & 9 \\
2 & 4 & 8 & 1 & 3 & 5 & 9 & 6 & 7 \\
\end{pmatrix} \in \omega S\tilde{\beta},
$$

$$
\omega_A = \begin{pmatrix}
1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 & 9 \\
1 & 2 & 5 & 3 & 4 & 6 & 7 & 8 & 9 \\
\end{pmatrix} \in S\tilde{\alpha}\omega S\tilde{\beta}.
$$

Also, $S\tilde{\alpha}\omega_A S\tilde{\beta} = S\tilde{\alpha}\omega_B S\tilde{\beta}$ implies $A = B$ since $a_{ij} = |\omega(J_j) \cap I_i|$ is constant on $S\tilde{\alpha}\omega S\tilde{\beta}$. 

Similarly, \((\omega_A, id^n)\) are representatives for \(S_{\tilde{\alpha}}[\Gamma]\backslash S_n[\Gamma]/S_{\tilde{\beta}}[\Gamma]\) by choosing appropriate \((\omega_{\tilde{\alpha}}, \gamma_{\tilde{\alpha}})\) and \((\omega_{\tilde{\beta}}, \gamma_{\tilde{\beta}})\).

For the third one, for any \(g \in GL_n\), we could choose appropriate \(k \in P_{\tilde{\beta}}, k' \in P_{\tilde{\alpha}}\) such that \(k\) and \(k'\) correspond to column operation and row operation that eliminate all terms but a permutation in \(g\).
Hence, there is a permutation representative \(\omega\) in all the double cosets. Since we already proved that there is \(\omega_A \in S_{\tilde{\alpha}} \omega S_{\tilde{\beta}}\), and \(S_{\tilde{\alpha}} < P_{\tilde{\alpha}}, S_{\tilde{\beta}} < P_{\tilde{\beta}}\), we can conclude that \(P_{\tilde{\alpha}} g P_{\tilde{\beta}}\) contains \(\omega_A\).

\[P_{\tilde{\alpha}} \omega_B P_{\tilde{\beta}} = P_{\tilde{\alpha}} \omega_B P_{\tilde{\beta}}\] implies \(A = B\) is similar.
To prove $\Delta \circ m = (m \otimes m) \circ (\text{id} \otimes T \otimes \text{id}) \circ (\Delta \otimes \Delta)$, it suffices to check that it holds in homogeneous component $(A \otimes A)_n$ i.e. for modules $U_1, U_2$ of $G_{r_1}, G_{r_2}$ with $r_1 + r_2 = n$, and $c_1 + c_2 = n$, we need

$$\text{res}^{n}_{c_1, c_2} \text{ind}^{n}_{r_1, r_2} (U_1 \otimes U_2) =$$

$$\bigoplus A \left( \text{ind}_{a_{11}, a_{21}}^{c_1} \otimes \text{ind}_{a_{12}, a_{22}}^{c_2} \right) \left( (\text{res}_{a_{11}, a_{12}}^{r_1} U_1 \otimes \text{res}_{a_{21}, a_{22}}^{r_2} U_2)^{-1} \right) \tau_A^{-1}$$

where $A = \begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix}$ has row sum $(r_1, r_2)$ and column sum $(c_1, c_2)$, and $\tau_A$ is the obvious isomorphism between $G_{a_{11}, a_{12}, a_{21}, a_{22}}$ and $G_{a_{11}, a_{21}, a_{12}, a_{22}}$. 
Mackey’s Formula

Consider subgroups $H, K < G$, and any $\mathbb{C}H$-module $U$. If $\{g_1, \ldots, g_t\}$ are double coset representatives for $K \backslash G/H$, then

$$\text{Res}_K^G \text{Ind}_H^G U \cong \bigoplus_{i=1}^t \text{Ind}_{g_i K \cap H}^K \left( \left( \text{Res}_{H \cap K g_i}^H U \right)^{g_i} \right)$$

where $g_i H = g_i H g_i^{-1}$ and $K^{g_i} = g_i^{-1} K g_i$. 
For the case $G_n = S_n$, applying Mackey’s formula with $G = G_n$, $H = G_{(r_1, r_2)}$, $K = G_{(c_1, c_2)}$, $U = U_1 \otimes U_2$ and double coset representatives $\{g_1, \ldots, g_t\} = \{\omega_{AT} \mid A \in \mathbb{N}^{2 \times 2}$ with row sum $(r_1, r_2)$ and column sum $(c_1, c_2)\}$, we get

$$\text{res}^n_{c_1, c_2} \text{ind}^n_{r_1, r_2}(U_1 \otimes U_2) =$$

$$\bigoplus_A \text{ind}^{G_{c_1, c_2}}_{\omega_{AT} (G_{r_1, r_2}) \cap G_{c_1, c_2}} \left( \left( \text{res}^{G_{r_1, r_2}}_{G_{r_1, r_2} \cap (G_{r_1, r_2})} \omega_{AT} U_1 \otimes U_2 \right)^{\omega_{AT}} \right).$$