Overview

What is a solid of revolution?

Method of Rings or Method of Disks

Method of Cylindrical Shells
What is a solid of revolution?

To get a solid of revolution we start out with a function \( y = f(x) \) on an interval \([a, b]\).

We then rotate this curve about a given axis to get the surface of the solid of revolution. Let's rotate the curve about the x-axis.

We want to determine the volume of the interior of this object.
How to calculate
**How to calculate**

1. We will first divide up the interval into n subintervals of width, $\Delta(x) = \frac{b-a}{n}$.
2. Then choose a point from each subinterval, $x_i^*$.
3. For volumes we will use disks on each subinterval to approximate the area. The area of the face of each disk is given by $A(x_i^*)$ and the volume of each disk is

$$V_i = A(x_i^*)\Delta(x).$$
The volume of the region can then be approximated by,

\[ V \approx \sum_{i=1}^{n} A(x_i^*) \Delta(x). \]

The exact volume is then,

\[ V = \lim_{n \to \infty} \sum_{i=1}^{n} A(x_i^*) \Delta(x) = \int_{a}^{b} A(x) \, dx. \]

So, in this case the volume will be the integral of the cross-sectional area at any \( x \), \( A(x) \).
In the sections where we actually use this formula we will also see that there are ways of generating the cross section that will actually give a cross-sectional area that is a function of $y$ instead of $x$. In these cases the formula will be,

$$= \int_{c}^{d} A(y)dy \quad c \leq y \leq d$$
Method of Rings or Method of Disks

One of the easier methods for getting the cross-sectional area is to cut the object perpendicular to the axis of rotation. Doing this the cross section will be either a solid disk if the object is solid (as our above example is) or a ring if we've hollowed out a portion of the solid (we will see this eventually).
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Method of Rings or Method of Disks

Finding the cross-sectional area
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\[ A = \pi (\text{radius})^2 \]

\[ A = \pi((\text{outer radius})^2 - (\text{inner radius})^2) \]
Method of disks or the method of rings

This Method in Four Steps:

1. Rotation about a horizontal axis \((y = y_0)\) → The cross sectional area \(A\) is a function of \(x\).
   Rotation about a vertical axis \((x = x_0)\) → The cross sectional area \(A\) is a function of \(y\) → rewrite the functions as a function of \(y\).

2. Find the area of \(A\), if it is a disk \(A = \pi (\text{radius})^2\), and if it is a ring \(A = \pi (\text{outer radius})^2 - (\text{inner radius})^2\).

3. Find the first occurrence of the ring at \(x = a\) (or \(y = c\)) and the last occurrence at \(x = b\) (or \(y = d\)).

4. Compute \(\int_a^b A(x)dx\) (or \(\int_c^d A(y)dy\)).
Example 1.1

Determine the volume of the solid obtained by rotating the region bounded by \( y = x^2 - 4x + 5 \), \( x = 1 \), \( x = 4 \) and the x-axis about the x-axis.

Solution.
$y = x^2 - 4x + 5$
The radius = the distance from the x-axis to the curve = the function value at that particular x as shown above. The cross-sectional area is then,

\[ A(x) = \pi(x^2 - 4x + 5)^2 = \pi(x^4 - 8x^3 + 26x^2 - 40x + 25) \]

Next we need to determine the limit of integration. Working from left to right the first cross section will occur at \( x = 1 \) and the last cross section will occur at \( x = 4 \). These are the limits of integration.

The volume of this solid is then,

\[ V = \int_{1}^{4} A(x) \, dx \]

\[ = \pi \int_{1}^{4} (x^4 - 8x^3 + 26x^2 - 40x + 25) \, dx \]

\[ = \pi \left( \frac{x^5}{5} - 2x^4 + \frac{26x^3}{3} - 20x^2 + 25x \right) \bigg|_{1}^{4} = \frac{78\pi}{5} \]
Example 1.2.

Determine the volume of the solid obtained by rotating the portion of the region bounded by \( y = \sqrt[3]{x} \) and \( y = \frac{x}{4} \) that lies in the first quadrant about the y-axis.

Solution.
This means that the inner and outer radius for the ring will be $x$ values and so we will need to rewrite our functions into the form $x = f(y)$. Here are the functions written in the correct form for this example.

\[
\begin{align*}
y &= \sqrt[3]{x} \implies x &= y^3 \\
y &= \frac{x}{4} \implies x &= 4y
\end{align*}
\]
The inner radius in this case is the distance from the y-axis to the inner curve while the outer radius is the distance from the y-axis to the outer curve. Both of these are then x distances and so are given by the equations of the curves as shown above. The cross-sectional area is then,

\[ A(y) = \pi ((4y)^2 - (y^3)^2) = \pi (16y^2 - y^6). \]

Working from the bottom of the solid to the top we can see that the first cross-section will occur at \( y = 0 \) and the last cross-section will occur at \( y = 2 \). These will be the limits of integration. The volume is then,

\[
V = \int_{c}^{d} A(y) \, dy = \pi \int_{0}^{2} (16y^2 - y^6) \, dy = \pi \left( \frac{16y^3}{3} - \frac{1}{7}y^7 \right) \Big|_{0}^{2} = \frac{512\pi}{21}.
\]
Example 1.3.

Determine the volume of the solid obtained by rotating the region bounded by $y = x^2 - 2x$ and $y = x$ about the line $y = 4$.

Solution.
inner radius = 4 - x

outer radius = 4 - (x^2 - 2x) = -x^2 + 2x + 4

The cross-sectional for this case is,

\[ A(x) = \pi \left( (-x^2 + 2x + 4)^2 - (4 - x)^2 \right) = \pi \left( x^4 - 4x^3 - 5x^2 + 24x \right) \]
The first ring will occur at \( x = 0 \) and the last ring will occur at \( x = 3 \) and so these are our limits of integration. The volume is then,

\[
\int_{a}^{b} A(x) \, dx
\]

\[
\pi \int_{0}^{3} x^4 - 4x^3 - 5x^2 + 24x \, dx
\]

\[
= \pi \left( \frac{1}{5}x^5 - x^4 - \frac{5}{3}x^3 + 12x^2 \right) \bigg|_{0}^{3} = \frac{153\pi}{5}
\]
Example 1.4.

Determine the volume of the solid obtained by rotating the region bounded by $y = 2\sqrt{x - 1}$ and $y = x - 1$ about the line $x = -1$.

Solution.

Now, let's notice that since we are rotating about a vertical axis and so the cross-sectional area will be a function of $y$. This also means that we are going to have to rewrite the functions to also get them in terms of $y$.

$$y = 2\sqrt{x - 1} \Rightarrow x = \frac{y^2}{4} + 1 \quad and \quad y = x - 1 \Rightarrow x = y + 1$$
outer radius = \( y + 1 + 1 = y + 2 \)

Inner radius = \( \frac{y^2}{4} + 1 + 1 = \frac{y^2}{4} + 2 \)

The cross-sectional area is then,

\[
A(y) = \pi \left( (y + 2)^2 - \left( \frac{y^2}{4} + 2 \right)^2 \right) = \pi \left( 4y - \frac{y^4}{16} \right)
\]
The first ring will occur at \( y = 0 \) and the final ring will occur at \( y = 4 \) and so these will be our limits of integration. The volume is,

\[
V = \int_c^d A(y) \, dy
\]

\[
= \pi \int_0^4 4y - \frac{y^4}{16}
\]

\[
= \pi \left( 2y^2 - \frac{1}{80}y^3 \right) \bigg|_0^4 = \frac{96\pi}{5}
\]
Method of Cylindrical Shells

In the previous section we only used cross sections that were in the shape of a disk or a ring. This however does not always need to be the case. We can use any shape for the cross sections as long as it can be expanded or contracted to completely cover the solid we are looking at.

**Example 2.1.** Determine the volume of the solid obtained by rotating the region bounded by \( y = (x - 1)(x - 3)^2 \) and the x-axis about the y-axis.
Why method of rings is not good in this case:

1. Both the inner and outer radius are defined by the same function

2. In general for a cubic polynomial to put a function in the form $x = f(y)$.

3. If we were to use rings the limit would be $y$ limits and this means that we will need to know how high the graph goes.
Method of Cylindrical Shells

The volume of the region can then be approximated by,

$$V \approx \sum_{i=1}^{n} A(x_i^*) \Delta(x).$$

The exact volume is then,

$$V = \lim_{n \to \infty} \sum_{i=1}^{n} A(x_i^*) \Delta(x) = \int_{a}^{b} A(x) \, dx.$$
Cross-Sectional area
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Method of Cylindrical Shells

This Method in Four Steps:

1. Rotation about a vertical axis \((x = x_0)\) → The cross sectional area \(A\) is a function of \(x\).
   (Rotation about a horizontal axis \((y = y_0)\) → The cross sectional area \(A\) is a function of \(y\) → rewrite the functions as a function of \(y\))

2. Find the area of \(A\), which is \(A = 2\pi(radius)(height)\) \((A = 2\pi(radius)(width))\).

3. Find the first cylinder which will cut into solid at \(x = a\) (or \(y = c\)) and the last cylinder which will cut into solid at \(x = b\) (or \(y = d\)). We do not take the complete range of \(x\) or \(y\) for the limits of integration Take a range of \(x\) or \(y\) that will cover one side of the solid.

4. Compute \(\int_a^b A(x)dx\) (or \(\int_c^d A(y)dy\)).
Example 2.1.

What we need to do is to find a different way to cut the solid that will give us a cross-sectional area that we can work with (cut with cylinders).

Doing this gives us a cylinder or shell in the object and we can easily find its surface area. The surface area of this cylinder is,

\[ A(x) = 2\pi (\text{radius})(\text{height}) = 2\pi (x) ((x - 1)(x - 3)^2) \]
The first cylinder will cut into solid at \( x = 1 \) and as we increase \( x \) to \( x = 3 \) we will completely cover both sides of the solid since expanding the cylinder in one direction will automatically expand it in the other direction as well. The volume of this solid is then,

\[
V = \int_{a}^{b} A(x) \, dx
\]

\[
= 2\pi \left[ \int_{1}^{3} x^4 - 7x^3 + 15x^2 - 9x \, dx \right]
\]

\[
= 2\pi \left( \frac{1}{5}x^5 - \frac{7}{4}x^4 + 5x^3 - \frac{9}{2}x^2 \right) \bigg|_{1}^{3} = \frac{24\pi}{5}
\]
Example 2.2

Determine the volume of the solid obtained by rotating the region bounded by \( y = \sqrt[3]{x} \), \( x = 8 \) and the x-axis about the x-axis.

Solution.

Rotating about a horizontal axis rewrite the function of the form \( x = f(y) \):

\[
y = \sqrt[3]{x} \Rightarrow x = y^3
\]
\[ A(y) = 2\pi (\text{radius})(\text{width}) \]
\[ = 2\pi (y)(8 - y^3) = 2\pi (8y - y^4) \]

The first cylinder will cut into the solid at \( y = 0 \) and the final cylinder will cut in at \( y = 2 \) and so these are our limits of integration.
The volume of this solid is,

\[ V = \int_c^d A(y) \, dy \]

\[ = 2\pi \int_0^2 8y - y^4 \, dy \]

\[ = 2\pi \left( \frac{4y^2}{2} - \frac{1}{5} \frac{y^5}{5} \right) \bigg|_0^2 \]

\[ = \frac{96\pi}{5} \]
Example 2.3

Determine the volume of the solid obtained by rotating the region bounded by \( y = 2\sqrt{x - 1} \) and \( y = x - 1 \) about the line \( x = 6 \).

Solution.

Rotating about a vertical axis \( A \) is a function of \( x \).
The cross sectional area is,

\[ A(x) = 2\pi (\text{radius})(\text{height}) = 2\pi (6 - x)(2\sqrt{x - 1} - x + 1) \]

Now the first cylinder will cut into the solid at \( x = 1 \) and the final cylinder will cut into the solid at \( x = 5 \) so there are our limits.
\[ V = \int_{a}^{b} A(x) \, dx \]

\[ = 2\pi \int_{1}^{5} (6 - x)(2\sqrt{x - 1} - x + 1) \]

\[ = 2\pi \int_{1}^{5} x^2 - 7x + 6 + 12\sqrt{x - 1} - 2x\sqrt{x - 1} \, dx \]

\[ = 2\pi \left( \frac{1}{3}x^3 - \frac{7}{2}x^2 + 6x + 8(x - 1)^{3/2} - \frac{4}{3}(x - 1)^{3/2} - \frac{4}{5}(x - 1)^{5/2} \right) \Bigg|_{1}^{5} \]

\[ = 2\pi \left( \frac{136}{15} \right) \]

\[ = \frac{272\pi}{15} \]
Example 2.4

Determine the volume of the solid obtained by rotating the region bounded by \( x = (y - 2)^2 \) and \( y = x \) about the line \( y = -1 \).

**Solution.** We should first get the intersection points there.

\[
y = (y - 2)^2 \Rightarrow (y - 4)(y - 1) = 0
\]

So, the two curves will intersect at \( y = 1 \) and \( y = 4 \).

Rotating about a horizontal axis, \( A \) is a function of \( y \).
Here is the cross sectional area for this cylinder.

\[ A(y) = 2\pi (\text{radius})(\text{width}) = 2\pi (y + 1) (y - (y - 2)^2) \]

The first cylinder will cut into the solid at \( y = 1 \) and the final cylinder will cut into the solid at \( y = 4 \).
\[ V = \int_c^d A(y) \, dy \]

\[ = 2\pi \int_1^4 2\pi (y + 1) (y - (y - 2)^2) \, dy \]

\[ = 2\pi \int_1^4 (-y^3 + 4y^2 + y - 4) \, dy \]

\[ = 2\pi \left( \left. \left( -\frac{1}{4} y^4 + \frac{4}{3} y^3 + \frac{1}{2} y^2 - 4y \right) \right|_1^4 \right) \]

\[ = \frac{63\pi}{2} \]
References

- James Stewart (2008)
  Calculus

- Paul’s Online Math Notes
The End