Dynamical Systems Methods in Early-Universe Cosmology

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• In our work, we relax the condition of isotropy to obtain cosmological models that admit Killing vectors that describe spatial translations only, and as a result are spatially homogeneous and anisotropic in general. Such models are known as the Bianchi models.
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- I will describe in this talk the method of **orthonormal frames** (Ellis and MacCallum, Comm. Math. Phys, 12, 108-141, 1969), which turn the Einstein field equations into a coupled system of nonlinear ordinary differential equations, and then describe the importance of numerical methods in understanding the global behaviour of the system.
Spacetime Splitting
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• In general relativity, \( n = 4 \), for three spatial coordinates, and one time coordinate.
Depiction of a manifold
Spacetime Splitting

General Relativity is described by the Einstein field equations:

\[ R_{ab} - \frac{1}{2} g_{ab} R + \Lambda g_{ab} = kT_{ab} \]

- \( R_{ab} \): Ricci tensor
- \( g_{ab} \): Metric tensor
- \( \Lambda \): Cosmological constant
- \( T_{ab} \): Energy-momentum tensor

These are 10 hyperbolic, nonlinear, coupled PDEs!
Spacetime Splitting

The solution of these equations is the metric tensor, typically written in the form:

\[ ds^2 = g_{ab} dx^a dx^b \]

One sees though, that the EFEs are not your “typical” field equations, in the sense that the solution is a metric tensor, and not a time-varying field, as would be the case with Maxwell’s equations, Navier-Stokes, etc…
Spacetime Splitting

To get a dynamical interpretation, then, we must **split** our spacetime into space and time. We accomplish this by **foliating** $M$ into a collection of spacelike hypersurfaces and allow a timelike vector to thread through them.

![Diagram showing foliation of spacetime with spacelike hypersurfaces at $t_1$, $t_2$, $t_3$, and $t_4$.]
In the standard $3+1$ decomposition, the metric is written as
\[ ds^2 = -\alpha^2 dt^2 + \gamma_{ij}(dx^i + \beta^i dt)(dx^j + \beta^j dt). \] (2.131)

Einstein's equations are then decomposed into the Hamiltonian constraint,
\[ R + K^2 - K_{ij}K^{ij} = 16\pi\rho, \] (2.132)
the momentum constraint,
\[ D_j(K^{ij} - \gamma^{ij}K) = 8\pi S^j, \] (2.133)
the evolution equation for the spatial metric,
\[ \partial_t\gamma_{ij} = -2\alpha K_{ij} + D_i\beta_j + D_j\beta_i \] (2.134)
(which is really a definition of the extrinsic curvature), and the evolution equation for the extrinsic curvature,
\[ \partial_t K_{ij} = \alpha(R_{ij} - 2K_{ik}K^k_j + KK_{ij}) - D_iD_j\alpha - 8\pi\alpha(S_{ij} - \frac{1}{2}\gamma_{ij}(S - \rho)) \\
+ \beta^k\partial_kK_{ij} + K_{ik}\partial_j\beta^k + K_{kj}\partial_i\beta^k. \] (2.135)
Useful contractions of the two evolution equations are
\[ \partial_t \ln \gamma^{1/2} = -\alpha K + D_i\beta^i \] (2.136)
and
\[ \partial_t K = -D^2\alpha + \alpha\left(K_{ij}K^{ij} + 4\pi(\rho + S)\right) + \beta^i D_iK, \] (2.137)
where we have used the Hamiltonian constraint (2.132) in deriving (2.137). The matter source terms appearing in the above equations are defined by
\[ \rho = n_a n_b T^{ab}, \quad S^i = -\gamma^{ij}n^a T_{aj}, \quad S_{ij} = \gamma_{ia}\gamma_{jb}T^{ab}, \quad S = \gamma^{ij}S_{ij}. \] (2.138)

(Baumgarte, Shapiro, 2010)
Energy-Momentum Tensor

The energy-momentum tensor is given by:

\[ T_{ab} = (\mu + p) u_a u_b + g_{ab} p - 3\xi H h_{ab} - 2\eta \sigma_{ab}, \]

In principle, other terms too, involving heat conduction, but these are **acausal**, (parabolic and elliptic PDEs do not occur in the real, physical universe, except in the sense of spacetime symmetries).

where \( \mu, p, \) and \( \sigma_{ab} \) denote the fluid’s energy density, pressure, and shear respectively, while \( \xi \) and \( \eta \) denote the bulk and shear viscosity coefficients of the fluid. Throughout this work, both coefficients are taken to be **nonnegative constants**. \( H \) denotes the Hubble parameter, and \( h_{ab} \equiv u_a u_b + g_{ab} \) is the standard projection tensor corresponding to the metric signature \((-,+,+,+,+))\).

Evolution and Constraint Equations

We will choose an orthonormal frame: \( \{ \mathbf{n}, \mathbf{e}_\alpha \} \)
That is, \( \mathbf{n} \) is tangent to a hypersurface-orthogonal congruence of geodesics, and we obtain:

The Einstein field equations:

\[
\begin{align*}
\dot{H} &= -H^2 - \frac{2}{3} \sigma^2 - \mu \left( \frac{1}{6} + \frac{1}{2} w \right), \\
\dot{\sigma}_{ab} &= -3H \sigma_{ab} + 2\epsilon_{(a \sigma_b)u} \Omega_v - S_{ab} - 2\eta \sigma_{ab}, \\
\mu &= 3H^2 - \sigma^2 + \frac{1}{2} R, \\
0 &= 3 \sigma^u a_u - \epsilon^{uv}_{a} \sigma^b_{u} n_{bv},
\end{align*}
\]
where $S_{ab}$ and $R$ are the three-dimensional spatial curvature and Ricci scalar and are defined as:

\[
S_{ab} = b_{ab} - \frac{1}{3} b^u u \delta_{ab} - 2 \epsilon^{uv} n_b u a_v, \quad (1)
\]

\[
R = -\frac{1}{2} b^u - 6 a_u a^u, \quad (2)
\]

where $b_{ab} = 2 n^u a n_{ub} - (n^u) n_{ab}$. We have also denoted by $\Omega_v$ the angular velocity of the spatial frame.

Using the Jacobi identities, one obtains evolution equations for these variables as well:

\[
\dot{n}_{ab} = -H n_{ab} + 2 \sigma^u (a n_b) u + 2 \epsilon^{uv} n_b u \Omega_v, \quad (1)
\]

\[
\dot{a}_a = -H a_a \sigma^b a_b + \epsilon^{uv} a_u \Omega_v, \quad (2)
\]

\[
0 = n^b a_b. \quad (3)
\]

The contracted Bianchi identities give the evolution equation for $\mu$ as

\[
\dot{\mu} = -3H (\mu + p) - \sigma^b a^a + 2 a_a q^a. \quad (4)
\]
In addition, we define a dimensionless “time” variable:

\[
\frac{dt}{d\tilde{\tau}} = \frac{1}{D}
\]

In our research, we are particularly interested in a closed universe, that is, of topology $S^3$.

The EFEs for such a universe take the form: (Kohli and Haslam, Phys. Rev. D. 89, 043518 (2014), arXiv: 1311.0389)
$$\tilde{H}' = -(1 - \tilde{H}^2)\tilde{q},$$

$$\tilde{\Sigma}'_+ = \tilde{\Sigma}_+ \tilde{H} (-2 + \tilde{q}) - 6\tilde{\Sigma}_+ \tilde{\eta}_0 - \tilde{S}_+,$$

$$\tilde{\Sigma}'_- = \tilde{\Sigma}_- \tilde{H} (-2 + \tilde{q}) - 6\tilde{\Sigma}_- \tilde{\eta}_0 - \tilde{S}_-,$$

$$\tilde{N}'_1 = \tilde{N}_1 \left( \tilde{H} \tilde{q} - 4\tilde{\Sigma}_+ \right),$$

$$\tilde{N}'_2 = \tilde{N}_2 \left( \tilde{H} \tilde{q} + 2\tilde{\Sigma}_+ + 2\sqrt{3}\tilde{\Sigma}_- \right),$$

$$\tilde{N}'_3 = \tilde{N}_3 \left( \tilde{H} \tilde{q} + 2\tilde{\Sigma}_+ - 2\sqrt{3}\tilde{\Sigma}_- \right),$$

$$\tilde{\Omega}' = \tilde{\Omega} \tilde{H} (-1 + 2\tilde{q} - 3w) + 9\tilde{H}^2 \tilde{\xi}_0 + 12\tilde{\eta}_0 \left( \tilde{\Sigma}^2_+ + \tilde{\Sigma}^2_- \right)$$

where

$$\tilde{q} = 2 \left( \tilde{\Sigma}^2_+ + \tilde{\Sigma}^2_- \right) + \frac{1}{2} \tilde{\Omega} (1 + 3w) - \frac{9}{2} \tilde{\xi}_0 \tilde{H}$$

$$\tilde{\Omega} + \tilde{\Sigma}^2 + \tilde{V} = 1$$

This is a constraint on the initial conditions.
Note that:

\[ \tilde{\mathcal{V}} = \frac{1}{12} \left[ \tilde{N}_1^2 + \tilde{N}_2^2 + \tilde{N}_3^2 - 2\tilde{N}_1\tilde{N}_2 - 2\tilde{N}_2\tilde{N}_3 - 2\tilde{N}_3\tilde{N}_1 + 3 \left( \tilde{N}_1\tilde{N}_2\tilde{N}_3 \right)^{2/3} \right] \]

\[ \tilde{S}_+ = \frac{1}{6} \left[ \left( \tilde{N}_2 - \tilde{N}_3 \right)^2 - \tilde{N}_1 \left( 2\tilde{N}_1 - \tilde{N}_2 - \tilde{N}_3 \right) \right], \]

\[ \tilde{S}_- = \frac{1}{2\sqrt{3}} \left[ \left( \tilde{N}_3 - \tilde{N}_2 \right) \left( \tilde{N}_1 - \tilde{N}_2 - \tilde{N}_3 \right) \right]. \]
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5. Investigate bifurcations that occur as per the parameter space
6. Knowing all this information allows one to state precisely information about the asymptotic past and future evolution of the universe model under consideration.
Examples of Fixed Points

\[ F_+ : \tilde{\Sigma}_+ = \tilde{\Sigma}_- = 0, \quad \tilde{N}_1 = \tilde{N}_2 = \tilde{N}_3 = 0, \]
\[ \tilde{H} = 1, \quad \tilde{\Omega} = 1. \]

Expanding Flat FLRW solution

Expanding Bianchi II solution

\[ \tilde{\Sigma}_+ = \frac{1}{16} \left[ 17 + 3w + 3\tilde{\eta}_0 + 9w\tilde{\eta}_0 - \gamma \right], \]
\[ \tilde{\Sigma}_- = 0, \]
\[ \tilde{N}_1 = \frac{1}{4} \sqrt{\frac{3}{2}} \left[ -3(63 - 38\tilde{\eta}_0 - 9\tilde{\eta}_0^2 + 3(w + 3w\tilde{\eta}_0)^2 - 2w(1 - 42\tilde{\eta}_0 + 9\tilde{\eta}_0^2)) + \gamma (13 + 3w - 9\tilde{\eta}_0 + 9w\tilde{\eta}_0 - 288\tilde{\xi}_0) \right]^{1/2}, \]
\[ \tilde{N}_2 = \tilde{N}_3 = 0, \quad \tilde{H} = 1, \]
\[ \tilde{\Omega} = \frac{1}{32} \left[ 15 - 3w - 54\tilde{\eta}_0 - 18w\tilde{\eta}_0 - 9\tilde{\eta}_0^2 - 27w\tilde{\eta}_0^2 + (1 + 3\tilde{\eta}_0)\gamma \right]. \]

Closed Einstein Static Universe

\[ F_c : \tilde{\Sigma}_{\pm} = 0, \quad \tilde{N}_1 = \tilde{N}_2 = \tilde{N}_3 = f > 0 \quad \mathbb{R}, \]
\[ \tilde{\Omega} = 1, \quad \tilde{\eta}_0 \geq 0. \]
<table>
<thead>
<tr>
<th>Equilibrium Point</th>
<th>EFE Solution</th>
<th>Local Sink</th>
<th>Local Source</th>
<th>Saddle</th>
</tr>
</thead>
<tbody>
<tr>
<td>F+</td>
<td>Expanding k=0 FLRW solution</td>
<td>Yes</td>
<td>No</td>
<td>Yes</td>
</tr>
<tr>
<td>F-</td>
<td>Contracting k=0 FLRW solution</td>
<td>Yes</td>
<td>Yes</td>
<td>Yes</td>
</tr>
<tr>
<td>P+(II)</td>
<td>Expanding Bianchi Type II Solution</td>
<td>No</td>
<td>No</td>
<td>Yes</td>
</tr>
<tr>
<td>P_(II) (new discovery!)</td>
<td>Contracting Bianchi Type II Solution</td>
<td>No</td>
<td>No</td>
<td>Yes</td>
</tr>
</tbody>
</table>
For nonlinear systems, we are bound by the Invariant Manifold theorem:

Let \( \mathbf{x} = \mathbf{0} \) be an equilibrium point of the DE \( \mathbf{x}' = \mathbf{f}(\mathbf{x}) \) on \( \mathbb{R}^n \) and let \( E^s, E^u, \) and \( E^c \) denote the stable, unstable, and centre subspaces of the linearization at \( \mathbf{0} \). Then there exists

- \( W^s \) tangent to \( E^s \) at \( \mathbf{0} \),
- \( W^u \) tangent to \( E^u \) at \( \mathbf{0} \),
- \( W^c \) tangent to \( E^c \) at \( \mathbf{0} \).

So, in nonlinear systems, linearization techniques only tell you the orbits of the dynamical system that belong to the stable, unstable, or centre manifolds.

So, we need global methods that will determine asymptotic stability of the various equilibrium points.
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  • Numerical methods

• The first three give information on the alpha and omega limit sets of the dynamical system.
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- In general, we need numerical methods to verify and check our topology-based work as outlined before, but, also, because of the dimension and nonlinearity of the dynamical system, there are situations in which a stability analysis will not yield any information, and where the global methods outline above will only give limited information.
Numerical Experiments

Parameter Space:

\[-1 \leq w \leq 1, \tilde{\eta}_0 \geq 0, \tilde{\xi}_0 \geq 0\]

Initial Conditions:

\[\tilde{\Omega} + \tilde{\Sigma}^2 + \tilde{V} = 1\]

Numerical Solver: ODE23s (MATLAB), ODE45 (MATLAB)

Solution:

\[\tilde{H}(\tau), \tilde{\Sigma}_\pm(\tau), \tilde{N}_\alpha(\tau), \tilde{\Omega}(\tau)\]
Chaotic Behaviour as $\tau \to -\infty$
This figure shows the dynamical system behavior for $\xi_0 = 0$, $\eta_0 = 0$, and $w = 1/3$. In particular, it displays the heteroclinic orbits joining $K_+$ to $K_-$, where $K_+$ is located at $\tilde{H} = 1$, and $K_-$ is located at $\tilde{H} = -1$ in the figure. Numerical solutions were computed for $-1000 \leq \tau \leq 1000$. For clarity, we have displayed solutions for shorter timescales.
This figure shows the dynamical system behaviour for $\tilde{\xi}_0 = 0$, $\tilde{\eta}_0 = 1/2$, and $\tilde{w} = -1/3$. The plus sign denotes the equilibrium point $F_+$. The model also isotropizes as can be seen from the last figure, where $\tilde{\Sigma}_\pm \to 0$ as $\tau \to \infty$. Numerical solutions were computed for $0 \leq \tau \leq 1000$. For clarity, we have displayed solutions for shorter timescales.
These figures show the dynamical system behavior for \( \hat{\xi}_0 = 0, \hat{\eta}_0 = 1/3, \) and \( w = 1. \) The plus sign denotes the equilibrium point \( F_- \). The model also isotropizes as can be seen from the last figure, where \( \hat{\Sigma}_\pm \to 0 \) as \( \tau \to \infty. \) Numerical solutions were computed for \( 0 \leq \tau \leq 1000. \) For clarity, we have displayed solutions for shorter timescales.
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When it is a saddle, the equilibrium point attracts along its stable manifold and repels along its unstable manifold. Therefore, some orbits will have an initial attraction to this point, but will eventually be repelled by it.

In the case when it was found to be a sink, all orbits approach the equilibrium point in the future. Therefore, there exists a time period and two separate configurations for which our cosmological model will isotropize and be compatible with present-day observations of a high degree of isotropy in the cosmic microwave background.