Practice Problems (Binary Operations)

1) **Background:** For $x,\, y \in \mathbf{R}$ define $x \simeq y$ to mean that $x^2 - 2x = y^2 - 2y$. You are given that $\simeq$ is an equivalence relation on $\mathbf{R}$.

Let E be the set of all equivalence classes of $\mathbf{R}$ for the relation $\simeq$; that is, $E = \{\, [x] \,|\, x \in \mathbf{R} \,\}$.

Extend addition on $\mathbf{R}$ to a binary operation "$\oplus$" on E defined by $[x] \oplus [y] = [x+y]$. For example $[3] \oplus [4] = [3+4] = [7]$.

(a) Display one other label for each of the equivalence classes [3] and [4].

(b) Use the equivalence classes [3] and [4] to demonstrate that "$\oplus$" is not a well-defined operation.

2) Let $\mathbf{R}^{\#}$ and $\mathbf{Q}^{\#}$ denote, respectively, the set of nonzero real numbers and the set of nonzero rational numbers. For $x,\, y \in \mathbf{R}^{\#}$ define $x \simeq y$ to mean that $x/y \in \mathbf{Q}^{\#}$. You are given that "$\simeq$" is an equivalence relation on $\mathbf{R}^{\#}$.
Let E be the set of all equivalence classes of $\mathbf{R}^{\#}$ for the relation $\simeq$; that is, $E = \{\, [x] \,|\, x \in \mathbf{R}^{\#} \,\}$. Extend the operation of multiplication from $\mathbf{R}^{\#}$ to E by defining $[x] \odot [y] = [xy]$.
Prove that the operation $\odot$ is well-defined.

3) In each of the following, prove or disprove that:

(i) $*$ is associative;

(ii) $*$ is commutative;

(iii) the given set contains an identity for $*$; and

(iv) if the set contains an identity for $*$, then each element in the set has an inverse in the set.

(a) For $x,\, y \in \mathbf{R}$, $x * y = y$.

(b) For $m,\, n \in \mathbf{N}$ (where $\mathbf{N}$ is the set of natural numbers), $m * n = 3^{mn}$.

(c) For $x,\, y \in \mathbf{R} - \{2\}$, $x * y = xy - 2x - 2y + 6$.

4) Let $*$ be an associative binary operation on a set S and let $e \in S$ be an identity for $*$.

(a) Prove that e is the unique identity of S for $*$.

(b) Suppose $a,\, b \in S$ are such that b is an inverse for a. Show that b is the unique inverse for a.

(c) Suppose that $x,\, y,\, z \in S$ are such that $x * y = e$ and $y * z = e$. Prove that $x = z$; hence x is the inverse of y.