How to have more things by forgetting where you put them

Mike Oliver

March 17, 2010
Two questions, not obviously related

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- In a picture like this:

  ![Diagram](image.png)

  can there ever be, in any sense, more purple squiggles than green ones?
Doubly infinite strings

The “things” I want to consider are doubly infinite strings on a finite alphabet, which might as well be the two letters 0 and 1:

\[ \ldots 1011101000100010100001 \ldots \]

Formally, these can be represented as maps from the integers to the set \( \{0, 1\} \); each string is a function \( f : \mathbb{Z} \to \{0, 1\} \).
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Or... can they?
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Or… can they?

Actually, given such an \(f\), there’s a particular value for \(f(0)\). So really one of these maps \(f\) is more like:

\[
\ldots 10111101000100010100001\ldots
\]

where \(f(0)\) is marked in red, \(f(1)\) is the letter to the right of the red one, \(f(-1)\) is the one to the left of the red one, and so on.
Forgetting the origin

So, if we really want to model doubly infinite strings, we have to forget where the origin is. That is, we want

\[ \ldots 111111111001110011111111111111 \ldots \]

with the 1s repeating infinitely in both directions, to be the same string as

\[ \ldots 111100111001111111111111111111 \ldots \]

again with 1s stretching out to infinity. The underlining is not meant to have any significance for the string — it’s just to make it visually clear why the two strings are the same. Or rather, would be the same, if we could forget which letter is red.
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The underlining is not meant to have any significance for the string — it’s just to make it visually clear why the two strings are the same. Or rather, would be the same, if we could forget which letter is red. Like so.
Forgetting is hard

Well, not really *hard*, but the formalities might not be obvious if you haven’t done them before.

- We’re pretty much stuck with functions $f : \mathbb{Z} \to \{0, 1\}$.
- But here’s the trick: We’ll say that two such functions are *equivalent* for our (momentary) purposes, if they are constant shifts of each other.
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- But here’s the trick: We’ll say that two such functions are *equivalent* for our (momentary) purposes, if they are constant shifts of each other.
- Given $f$ and $g$, if, say, you can shift $f$ by three spaces and get $g$; that is, $f(n + 3) = g(n)$ for every integer $n$, then $f$ and $g$ are equivalent, $f \sim g$. Or, any other integer in place of the 3.
- Then, formally, our objects are the quotient of the functions from $\mathbb{Z}$ to $\{0, 1\}$, modulo $\sim$. 
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- Then, formally, our objects are the quotient of the functions from $\mathbb{Z}$ to $\{0, 1\}$, modulo $\sim$.
- But mostly we won’t get that formal. We’ll just require that all “well-defined” questions about doubly infinite strings, *must give the same answer* for $f$ as for $g$, if $f$ and $g$ are shifts of one another.
What does “more” mean?

After all, we’re talking about infinite sets. Aren’t all infinities the same?
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- For example, there are exactly as many rational numbers as natural numbers ($\aleph_0$). Both sets are *countably infinite*. 
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• For example, there are exactly as many rational numbers as natural numbers ($\aleph_0$). Both sets are countably infinite.

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These facts can be seen via very simple arguments due to Georg Cantor, which you have all likely seen.
OK, so what does it mean?

- To claim that set $A$ has fewer elements than (or the same number as) set $B$, in symbols
  
  $$|A| \leq |B|$$
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  we need a map that sends every element of $A$ to a *unique* element of $B$. 
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• “Unique” meaning that if you have two different things in $A$, their arrows don’t collide on the $B$ side.
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That is, you can’t do this.
OK, so what does it mean?
(cont.)

- The map doesn’t have to be a *rule* of any sort. It can just be an arbitrary bunch of arrows.
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However, we *are* sometimes interested in whether there’s a “reasonably definable” rule saying where the arrows go. (If we weren’t, this talk would be very short.)
Probabilities, with origin

Suppose we make a doubly infinite string by flipping a coin. Heads, we set $f(0)$ to 1; tails, to 0. Flip again to get $f(1)$, then again for $f(-1)$, then $f(2)$, $f(-2)$, and so on:
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\[
\ldots \quad 011 \quad \ldots \quad \text{TAILS}
\]
Intro

What am I talking about?

All or nothing

How many strings

Conclusion

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\ldots \quad 1010111000 \quad \ldots
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What’s the probability that:

- The red digit — that is, \( f(0) \) — is 0?
- The red digit and the next two are 111?
- At least one of the two digits immediately after the red one, is 0?
- \ldots and so on.
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\ldots and so on.
Probabilities, forgetting origin

Suppose we have the same coin-flipping setup, but we forget which digit is red. Now we can’t ask any of the probability questions on the previous slide. Are there any probability questions we can ask?

Certainly. We can ask any question whose answer will always be the same for $f$ and $g$ such that $f \sim g$; questions that give the same answer if you shift the string. What's the probability that:

- There's at least one 0 in the string?
- The average density of 0s in the string is greater than $\frac{2}{3}$?
- The entire text of *Hamlet* appears in the string?

As it turns out, for any "reasonably definable" probability question we can ask about the forgotten-origin string, the answer is always exactly 0 or exactly 1. (This is called **ergodicity**.)
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- There’s at least one 0 in the string? $1$
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- There’s at least one 0 in the string? 1
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How many doubly-infinite strings?

It’s easy to count the number of doubly-infinite strings. There are exactly as many as there are singly-infinite strings; namely, $2^{\aleph_0}$. Remember that to show that the number of doubly-infinite strings is $\leq$ the number of singly-infinite strings, we just have to find a map that converts a doubly-infinite string to a unique singly-infinite string.
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\[
\begin{array}{cccc}
\vdots & 1 & \vdots \\
1 & & \vdots \\
\end{array}
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110 ...
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... 101110 ...
110110 ...
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\[ \ldots \quad 101011100 \quad \ldots \]
\[ 110110001 \quad \ldots \]
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\[
\begin{align*}
\ldots & \quad 1010111000 \quad \ldots \\
1101100010 & \\
\ldots & 
\end{align*}
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$$\ldots \ 1010111000 \ \ldots$$

$$1101100010$$

$$\ldots$$

$\ldots$ but of course this requires knowing which letter of the doubly-infinite string is red.
So what if we don’t know the origin?

- If we don’t have origins for the doubly-infinite strings, then we can’t use the trick of the previous slide to assign a unique singly-infinite string to each doubly-infinite string, because we don’t know where to start. But maybe it can still be done, just in some more complicated way.
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- But if $f$ is not a shift of $g$, then $F(f) \neq F(g)$ (that’s the uniqueness).
Revisiting cardinality, for sets of equivalence classes

- In this picture, the number of equivalence classes (dotted ovals) in $A$ is $\leq$ the number of points in $B$
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- But green dots in different ovals must match to different blue dots.
Is there such a map?

- Yes, there is.
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However this doesn’t appear to help us if we want a “reasonably definable” such map. The axiom just tells us that there *is* a way of distinguishing one string in each box; it doesn’t tell us how to do it. Of course, it also doesn’t tell us that there *isn’t* a definable way to do it.
Tying things together (with strings)

So suppose we *do* have a “reasonably definable” map $F$ that shows that there are only as many forgotten-origin doubly-infinite strings as there are singly-infinite strings. What can we find out about it? For example, we might want to know, if you put a doubly-infinite string $s$ into $F$, giving a singly-infinite string $F(s)$, does $F(s)$ start with a 1? Of course, we expect the answer to depend on $s$....
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But! That’s a “reasonably definable” question.

So the probability that $F(s)$ starts with 1, for a *random* string $s$, is either exactly 0 or exactly 1.
Building a special string, based on $F$

So now let's look at the probability that the $n$th letter of $F(s)$ is 1 (for a random doubly-infinite string $s$)
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Probability that bit number 0 is 1:
Special string: 0 ...
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Probability that bit number 1 is 1:
Special string: 00 $\ldots$
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Probability that bit number 2 is 1:

Special string: $001\ldots$
Building a special string, based on $F$

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Probability that
bit number 3 is 1:
Special string: 0010 ...
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Probability that bit number 4 is 1:
Special string: 00101 ...
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Special string: 001011 ...
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What's the probability that

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- Because for each $n$, the probability that the $n$th bit of $F(s)$, is the $n$th bit of the special string, is 1, by construction
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  - Because for each $n$, the probability that the $n^{th}$ bit of $F(s)$, is the $n^{th}$ bit of the special string, is 1, by construction
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• Almost everything on the left, gets mapped to the special string.
Picture of $F$

- Again, the green dots are strings-with-origin; the ovals around them gather them into strings-without-origin.
- The blue dots are singly-infinite strings — one of them is the “special string”.
- *Almost everything* on the left, gets mapped to the special string.
- So a random green dot has probability one of being in the big oval.
Is this possible?

A

B

special string
Is this possible?

- No

- No

- Each individual green dot has probability zero

- But there are only countably many green dots in an oval (because there are only countably many amounts by which you can shift a string)

- And probability is countably additive

So we have a contradiction on our hands. The only way to resolve it is that no such function $F$ exists.
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This says that there are more boxes, than there are total things in all the boxes put together! Even though each box has infinitely many things in it.
In the picture, there are strictly more purple guys than green guys!
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This says that there are more boxes, than there are *total* things in all the boxes put together! Even though each box has infinitely many things in it.

In the picture, there are strictly more purple guys than green guys! (Well, at least in this special sense, where we require definable maps.)
Wait a minute, did we exactly show *more*?

Well, no, not quite. We showed that the number of forgotten-origin strings was, in this special sense, *not less than or equal to* the number of strings with origin:

$$|\{\text{no-origin strings}\}| \not\leq_{\text{definable}} |\{\text{strings with origin}\}|$$
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- That is, we need a definable one-to-one map $F$ from strings-with-origin to strings-without-origin
- In this context, one-to-one means that if $s_1$ is different from $s_2$, then $F(s_1)$ is not only different from $F(s_2)$; it’s not even a shift of $F(s_2)$
How do we get that map?

I’m actually going to let you think about that one. It’s a really good problem to check your understanding. But here’s a couple of hints:
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• So just to get started, can you think of a way to code the first bit of the singly-infinite string? Maybe, if it’s zero, you make a doubly-infinite string that has lots of zeroes, and if it’s a one, you make a doubly-infinite string that has lots of ones? Can you generalize?
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- Alternatively, you might first code the singly-infinite string as a real number, then try to encode the real number into the doubly-infinite string, in such a way that you can decode it after any shift.
Opportunities for research

- The example I have shown today is a small sample of a very active research field, usually called the study of Borel equivalence relations.
Opportunities for research

- The example I have shown today is a small sample of a very active research field, usually called the study of *Borel equivalence relations*.
- If you are interested in mathematical logic, this is a newer subfield of set theory than many of the traditional ones, and may have more accessible open problems.
Partial chart of equivalence relations, ordered by definable cardinality of quotient