Kleisli enriched

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Abstract

For a monad S on a category $\mathscr{K}$ whose Kleisli category is a quantaloid, we introduce the notion of modularity, in such a way that morphisms in the Kleisli category may be regarded as V-(bi)modules (= profunctors, distributors), for some quantale V. The assignment $S \longmapsto V$ is shown to belong to a global adjunction which, in the opposite direction, associates with every (commutative, unital) quantale V the prototypical example of a modular monad, namely the presheaf monad on V-**Cat**, the category of (small) V-categories. We discuss in particular the question whether the Hausdorff monad on V-**Cat** is modular.

Keywords: modular monad, Kleisli category, quantale, quantaloid, V-(bi)module, V-category, power-set monad, presheaf monad, Hausdorff monad.
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1. Introduction

For a monad S on a category $\mathscr{K}$, a morphism in the Kleisli category of S is given by a morphism of type $Y \longrightarrow SX$ in $\mathscr{K}$. As the carrier of the free Eilenberg-Moore algebra over X, naturally SX carries additional structure which may be inherited by the relevant hom-set of the Kleisli category. For example, when S is the power-set monad on **Set**, so that $SX = PX$ is the free sup-lattice over the set X, the Kleisli category is the (dual of the) category of sets and relations and, hence, a quantaloid, i.e., a **Sup**-enriched category. Less trivially, and more generally, taking presheaves over a (small) category X defines a monad P on **Cat** whose Kleisli category is the (bi)category of categories and bimodules (= profunctors, distributors), the rich structure of which is a fundamental tool for a substantial body of categorical research. We refer the reader in particular to [12], [6], [5] and [4], and the extensive lists of references in these papers which point the reader also to the origins of a theme that seems to have interested researchers for some forty years. Some of these papers consider the presheaf monad in the enriched context (see [8]), i.e., for V-**Cat**, where V is a symmetric monoidal-closed category (rather than the classical $V =$ **Set**), or even a bicategory (see [10], for example). The more manageable case when the bicategory is just a quantaloid has been considered by Stubbe [13] who exhibits the passage from V to the category of V-bimodules as a morphism of quantaloids.

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In this paper we consider the further simplified case when V is a quantale, i.e., a one-object quantaloid which, based on Lawvere's treatment of metric spaces [9] and Barr's presentation of topological spaces [2], has been used to set up a common syntax for various categories of interest in analysis and topology; see, for example, [3], [11], [7], [14]. Specifically, with (the dual of) the Kleisli category of the presheaf monad P on V-**Cat** describing precisely the quantaloid V-**Mod** of V-categories and V-modules, we ask ourselves the question when the morphisms of the Kleisli category of an arbitrary monad $\mathbb{S}$ on a category may be treated as V-modules, for some quantale V. To this end we introduce the notion of a *modular monad* on an abstract category $\mathscr{K}$ which asks its Kleisli category to be a quantaloid (and, hence, a 2-category) in which $\mathscr{K}$-morphisms have adjoints. By means of a distinguished "unital" object E in $\mathscr{K}$ one may then associate with the monad $\mathbb{S}$ a quantale V and establish a fully faithful "comparison functor" from the (dual of the) Kleisli category to V-**Mod**. With some natural restrictions on both the objects and morphisms, this fully faithful functor plays the role of a unit of an adjunction between a (very large) category of modular monads and the category of (commutative unital) quantales, the counits of which are isomorphisms. In other words, assigning to every V the presheaf monad on V-**Cat** defines, up to categorical equivalence, a full reflective embedding of the category of quantales into a category of modular monads. In setting up this category, some care must be given to the definition of its morphisms since the assignments $\mathbb{S} \longmapsto V$ and $V \longmapsto \mathbb{P}$ behave surprisingly crudely with respect to the natural 2-categorical structures of these categories. We have therefore used 2-cells only to the minimal extent necessary to answer our original question.

The paper is written in a largely self-contained style; it therefore recalls some known facts, giving sufficient details in particular on our prototypical example of a modular monad, the presheaf monad on V-**Cat** (Section 2). Having already set up a "modular terminology" for the dual of the Kleisli category of a modular monad in that introductory section, in Section 3 we prove the comparison theorem with the modules of actual V-categories, for some suitable V. Section 4 contains a detailed discussion of the question to which extent the Hausdorff monad (see [1]) is modular. Finally, global correspondences between quantales and modular monads are established in Sections 5 and 6, first in terms of functors that are partially (pseudo-)inverse to each other, and then in terms of the somewhat suprising adjunction that exhibits the ordinary category of quantales as a very substantial part of a very large environment of (certain) modular monads.

2. Modular monads

Let $\mathbb{S} = (S, \varepsilon, \nu)$ be a monad on a category $\mathscr{K}$. We denote the opposite of the Kleisli category of $\mathbb{S}$ by

$$\mathbb{S}\text{-}\mathbf{Mod}\,.$$

Hence, its objects are the objects of $\mathscr{K}$, and a morphism $\varphi\colon X \longrightarrow\!\!\!\!\!\circ\; Y$, also called $\mathbb{S}$-*module* from X to Y, is given by a $\mathscr{K}$-morphism $\varphi\colon Y \longrightarrow SX$; composition with $\psi\colon Y \longrightarrow\!\!\!\!\!\circ\; Z$ is defined by

$$\psi \circ \varphi := \nu_X \cdot S\varphi \cdot \psi,$$

and $1_X^* := \varepsilon_X : X \longrightarrow\!\!\!\!\!\circ\; X$ is the identity morphism on X in $\mathbb{S}$-**Mod**. Extending this notation, one has the right-adjoint functor

$$(-)^*\colon \mathscr{K}^{\mathsf{op}} \longrightarrow \mathbb{S}\text{-}\mathbf{Mod}$$

which sends $f\colon X \longrightarrow Y$ in $\mathscr{K}$ to $f^* := \varepsilon_Y \cdot f\colon Y \longrightarrow\!\!\!\!\circ\; X$ in S-**Mod**. We denote its left adjoint by

$$\widehat{(-)}\colon S\text{-}\mathbf{Mod} \longrightarrow \mathscr{K}^{\mathsf{op}},$$

sending X to SX and $\varphi\colon X \longrightarrow\!\!\!\!\circ\; Y$ to $\hat{\varphi} := \nu_X \cdot S\varphi\colon SY \longrightarrow SX$ in $\mathscr{K}$. The adjunction produces two factorizations of φ, namely

$$\varphi = \hat{\varphi} \cdot \varepsilon_Y \quad \text{in } \mathscr{K} \qquad \text{and} \qquad \varphi = \varphi^* \circ \iota_X \quad \text{in } S\text{-}\mathbf{Mod},$$

with the morphisms ε_X serving as counits (in $\mathscr{K}^{\mathsf{op}}$), and the morphisms $\iota_X := 1_{SX}\colon X \longrightarrow\!\!\!\!\circ\; SX$ as units (in S-**Mod**). Units and counits are connected by the triangular equalities (which are special cases of the factorization)

$$\widehat{\iota_X} \cdot \varepsilon_{SX} = 1_{SX} \quad \text{in } \mathscr{K} \qquad \text{and} \qquad \varepsilon_X^* \circ \iota_X = 1_X^* \quad \text{in } S\text{-}\mathbf{Mod}.$$

Definition 2.1. *We call the monad S on $\mathscr{K}$* ***modular*** *if*

1. S-**Mod** *carries the structure of a quantaloid, that is: every* hom-*set carries the structure of a complete lattice, such that composition in* S-**Mod** *from either side preserves arbitrary suprema;*
2. *for every morphism $f\colon X \longrightarrow Y$ in $\mathscr{K}$, $f^*\colon Y \longrightarrow\!\!\!\!\circ\; X$ has a left adjoint in the 2-category* S-**Mod**, *that is: there exists $f_*\colon X \longrightarrow\!\!\!\!\circ\; Y$ in* S-**Mod** *with $1_X^* \leq f^* \circ f_*$ and $f_* \circ f^* \leq 1_Y^*$;*
3. *there is an object E in $\mathscr{K}$ with $\mathscr{K}(E, E) = \{1_E\}$ and*

$$\bigvee_{x\colon E \to X} x_* \circ x^* = 1_X^*$$

for all X in $\mathscr{K}$.

For a modular monad S, we always fix the order that makes S-**Mod** a quantaloid and assume a fixed choice of the left adjoints f_* and the distinguished object E; in other words, modularity is not considered as a property of the monad, but as a structure on it.

Example 2.2. For the power-set monad $(P, \{-\}, \bigcup)$ on **Set**, when one considers maps $\varphi\colon Y \longrightarrow PX$ as relations $\varphi\colon X \longrightarrow\!\!\!\!\circ\; Y$ (by writing $x\ \varphi\ y$ instead of $x \in \varphi(y)$), P-**Mod** is simply the category **Rel** of sets and relations. Its hom-sets inherit the inclusion order of power-sets, which makes P-**Mod** a quantaloid. One takes $f_*\colon Y \longrightarrow PX$ to be the inverse-image function of $f\colon X \longrightarrow Y$ in **Set**, and with E a singleton set, condition 3 of 2.1 amounts to the trivial statement

$$(\exists x \in X : u = x \,\&\, x = v) \iff u = v$$

for all $u, v \in X$. Hence, P is modular.

Example 2.3. Replacing the 2-element chain in $PX = 2^X$ by an arbitrary frame V, one generalizes the previous example, as follows. For $f\colon X \longrightarrow Y$ in **Set**, $P_V f\colon P_V X = V^X \longrightarrow P_V Y$ be left adjoint to $V^f\colon V^Y \longrightarrow V^X, \beta \longmapsto \beta \cdot f$; hence

$$(P_V f)(\alpha)(y) = \bigvee_{x \in f^{-1}(y)} \alpha(x),$$

for all $\alpha\colon X \longrightarrow V$, $y \in Y$. The maps

$$x \xrightarrow{\delta_X} P_V X \qquad \text{and} \qquad P_V P_V X \xrightarrow{\nu_X} P_V X$$

with $\delta_X(x)(x') = \top$ (the top element in V) if $x = x'$ and $\bot$ (bottom) else, and with

$$\nu_X(\Sigma)(x) = \bigvee_{\alpha \in P_V X} \Sigma(\alpha) \wedge \alpha(x)$$

for all $\Sigma\colon V^X \longrightarrow V$, $x, x' \in X$, give P_V the structure of a monad. Using P instead of P_V, we can describe P-**Mod** equivalently as the quantaloid V-**Rel** of sets with V-valued relations $\varphi\colon X \longrightarrow\!\!\!\!\!\circ\;\, Y$ as morphisms. Indeed, maps $\varphi\colon Y \longrightarrow PX = V^X$ correspond bijectively to maps $\tilde{\varphi}\colon X \times Y \longrightarrow V$, and composition in P-**Mod** becomes the ordinary composition of V-valued relations:

$$\widetilde{\psi \circ \varphi}(x,z) = (\tilde{\psi} \circ \tilde{\varphi})(x,z) = \bigvee_{y \in Y} \tilde{\psi}(y,z) \wedge \tilde{\varphi}(x,y),$$

for $\psi\colon Y \longrightarrow\!\!\!\!\!\circ\;\, Z$, $x \in X$, $z \in Z$. The left adjoint f_* of f^* for $f\colon X \longrightarrow Y$ is obtained by interchanging variables:

$$f_*(y)(x) = f^*(x)(y) = \top$$

if $f(x) = y$, and $\bot$ else. Condition 3 of 2.1 is, as in the case $V = 2$, trivially satisfied also in general. Consequently, $P = P_V$ is a modular monad on **Set**.

Example 2.4. Let V be a unital quantale (= one-object quantaloid), i.e. a complete lattice with a binary associative operation $\otimes$ and a neutral element k such that $\otimes$ preserves suprema in each variable. (Every frame V as in 2.3 is a quantale, with $\otimes = \wedge$, $k = \top$.) The category V-**Cat** of (small) V-categories and V-functors has as objects sets X which come with a function $X \times X \longrightarrow V$ (whose value on (x,y) we denote by $X(x,y)$) such that

$$k \le X(x,x) \qquad \text{and} \qquad X(y,z) \otimes X(x,y) \le X(x,z);$$

morphisms $f\colon X \longrightarrow Y$ satisfy $X(x,y) \le Y(f(x), f(y))$, for all $x, y, z \in V$. The quantale V itself is a V-category, with the V-category structure $V(v,w) = v \multimap w$ given by its own "internal hom" defined by

$$z \le v \multimap w \iff z \otimes v \le w$$

for all $z, v, w \in V$. Moreover, V-**Cat** has an "internal hom" with

$$X \multimap Y = V\text{-}\mathbf{Cat}(X,Y) \qquad \text{and}$$
$$(X \multimap Y)(f,g) = \bigvee_{x \in X} Y(f(x), g(x)).$$

When V is commutative, V-**Cat** is symmetric monoidal closed, with $X \otimes Y = X \times Y$ and

$$(X \otimes Y)((x,y),(x',y')) = X(x,x') \otimes Y(y,y'),$$

and one can also form the opposite X^{op} of a V-category X, with $X^{\mathrm{op}}(x,y) = X(y,x)$. Now, the Yoneda embedding

$$\mathfrak{y}_X\colon X \longrightarrow P_V^{\circ} X := (X^{\mathrm{op}} \multimap V), \qquad x \longmapsto X(-,x),$$

provides the unit of the presheaf monad $(P_V^\circ, \mathfrak{y}, \mathfrak{m})$ of V-**Cat**, as follows. Writing P instead of P_V°, for $f\colon X \longrightarrow Y$, the V-functor $Pf\colon PX \longrightarrow PY$ is defined by

$$(Pf)(\alpha)(y) = \bigvee_{x\in X} Y(y, f(x)) \otimes \alpha(x)$$

for all $\alpha \in PX$, $y \in Y$, and the monad multiplication $\mathfrak{m}_X\colon PPX \longrightarrow PX$ is given by

$$\mathfrak{m}_X(\Sigma)(x) = \bigvee_{\alpha\in PX} \Sigma(\alpha) \otimes \alpha(x).$$

We claim that *the category P-**Mod** is precisely the category V-**Mod** whose objects are V-categories, and whose morphisms $\varphi\colon X \longrightarrow\!\!\!\!\circ\; Y$ are V-(bi)modules, also called V-distributors or V-profunctors*, given by functions $\varphi\colon X \times Y \longrightarrow V$ satisfying

$$Y(y_1, y_2) \otimes \varphi(x_2, y_1) \otimes X(x_1, x_2) \leq \varphi(x_1, y_2) \tag{$*$}$$

for all $x_1, x_2 \in X$, $y_1, y_2 \in Y$; composition with $\psi\colon Y \longrightarrow\!\!\!\!\circ\; Z$ is defined by

$$(\psi \circ \varphi)(x, z) = \bigvee_{y\in Y} \psi(y, z) \otimes \varphi(x, y). \tag{$**$}$$

Proof. Since $(*)$ is equivalent to $\varphi\colon X^{\mathrm{op}} \otimes Y \longrightarrow V$ being a V-functor we may as well think of φ as a V-functor $Y \longrightarrow PX$, writing $\varphi(x, y)$ as $\varphi(y)(x)$. Hence, all we need to verify is that the composition $(**)$ in V-**Mod** coincides with the Kleisli composition of P-**Mod**, i.e.,

$$\bigvee_{y\in Y} \varphi(y)(x) \otimes \psi(z)(y) = (\mathfrak{m}_X \cdot P\varphi \cdot \psi)(z)(x) \tag{$***$}$$

for all $\varphi\colon Y \longrightarrow PX$, $\psi\colon Z \longrightarrow PY$ in V-**Cat**, $x \in X$, $z \in Z$. By the Yoneda Lemma the left-hand side of $(***)$ may be rewritten and compared to the right-hand side, as follows:

$$\begin{aligned}\bigvee_{y\in Y} PX(\mathfrak{y}_X(x), \varphi(y)) \otimes \psi(z)(y) &= (P\varphi)(\psi(z))(\mathfrak{y}_X(x))\\ &\leq (P\varphi)(\psi(z))(\mathfrak{y}_X(x)) \otimes \mathfrak{y}_X(x)(x)\\ &\leq \bigvee_{\alpha\in PX} (P\varphi)(\psi(z))(\alpha) \otimes \alpha(x)\\ &= \mathfrak{m}_X\big((P\varphi)(\psi(z))\big)(x).\end{aligned}$$

For "$\geq$", since

$$\Big(\bigvee_{x'\in X} \alpha(x') \multimap \varphi(y)(x')\Big) \otimes \alpha(x) \leq (\alpha(x) \multimap \varphi(y)(x)) \otimes \alpha(x) \leq \varphi(y)(x),$$

one has, for all $\alpha \in PX$,

$$\begin{aligned}(P\varphi)(\psi(z))(\alpha) \otimes \alpha(x) &= \bigvee_{y\in Y} PX(\alpha, \varphi(y)) \otimes \psi(z)(y) \otimes \alpha(x)\\ &\leq \bigvee_{y\in Y} \varphi(y)(x) \otimes \psi(z)(y),\end{aligned}$$

as desired. □

With suprema formed pointwise (as in V-**Rel** of 2.3), V-**Mod** becomes a quantaloid, and it is easy to check the remaining conditions to confirm that *P is modular*, by putting $f_*(y)(x) := Y(f(x), y)$ for all $x \in X$, $y \in Y$, $f\colon X \longrightarrow Y$ in V-**Cat**.

Note that Example 2.3 is a special case of 2.4, by restriction to *discrete* V-categories, i.e., sets.

In Section 4 we discuss an example showing that conditions 2, 3 of 2.1 do not follow from condition 1 of 2.1.

3. Exhibiting Kleisli morphisms as *V*-modules

The following theorem shows that Example 2.4 exhibits the prototypical modular monad, and it justifies the module terminology for the morphisms of its Kleisli category.

Theorem 3.1. *Let $S = (S, \varepsilon, \nu)$ be a modular monad on a category $\mathscr{K}$. Then there is a unital quantale V and a functor $|{-}|\colon \mathscr{K} \longrightarrow V$-**Cat** that can be lifted to a full and faithful homomorphism $|{-}|\colon S$-**Mod** $\longrightarrow V$-**Mod** of quantaloids such that*

$$\begin{array}{ccc} S\text{-}\mathbf{Mod} & \xrightarrow{\;|-|\;} & V\text{-}\mathbf{Mod} \\ \uparrow{\scriptstyle (-)^*} & & \uparrow{\scriptstyle (-)^*} \\ \mathscr{K}^{\mathrm{op}} & \xrightarrow[\;|-|^{\mathrm{op}}\;]{} & (V\text{-}\mathbf{Cat})^{\mathrm{op}} \end{array}$$

commutes.

Proof. The monoid structure of $V := S\text{-}\mathbf{Mod}(E, E) = \mathscr{K}(E, SE)$ makes V a unital quantale since S-**Mod** is a quantaloid. The hom-functor $|{-}| = \mathscr{K}(E, -)$ takes values in V-**Cat** if, for a $\mathscr{K}$ object X, we put

$$|X|(x, y) := y^* \circ x_*$$

for all $x, y \in |X|$. Indeed, the adjunctions $x_* \dashv x^*$ show the V-category laws for $|X|$, and for a morphism $f\colon X \longrightarrow Y$ in $\mathscr{K}$ one obtains the V-functoriality of $|f|$ from

$$|X|(x, y) = y^* \circ 1_X^* \circ x_* \leq y^* \circ f^* \circ f_* \circ x_* \leq (f{\cdot}y)^* \circ (f{\cdot}x)_* = |Y|(|f|(x), |f|(x))$$

for all $x, y \in |X|$; here we used the fact that the local adjunctions $f_* \dashv f^*$ compose, so that functoriality of $(-)^*$ produces a pseudofunctor $(-)_*\colon \mathscr{K} \longrightarrow S$-**Mod**. For $\varphi\colon X \longrightarrow\!\!\circ\; Y$ in S-**Mod** one defines $|\varphi|\colon |X| \longrightarrow |Y|$ in V-**Mod** by

$$|\varphi|(x, y) = y^* \circ \varphi \circ x_*$$

for $x \in |X|$, $y \in |Y|$; V-modularity follows immediately from the local adjunctions. Also, for $f\colon X \longrightarrow Y$ in $\mathscr{K}$, one has

$$|f^*|(y, x) = x^* \circ f^* \circ y_* = (f{\cdot}x)^* \circ y_* = |Y|(y, |f|(x)) = |f|^*(y, x),$$

for all $x \in |X|, y \in |Y|$. Next we show that $|{-}|\colon S\text{-}\mathbf{Mod} \longrightarrow V\text{-}\mathbf{Mod}$ preserves composition and suprema. In fact, for $\psi\colon Y \longrightarrow\!\!\!\!\circ\; Z$ and $x \in |X|, z \in |Z|$ one has:

$$\begin{aligned} |\psi \circ \varphi|(x,z) &= z^* \circ \psi \circ 1_Y^* \circ \varphi \circ x_* \\ &= z^* \circ \psi \circ \Big(\bigvee_{y\in|Y|} y_* \circ y^*\Big) \circ \varphi \circ x_* \\ &= \bigvee_{y\in|Y|} (z^* \circ \psi \circ y_*) \circ (y^* \circ \varphi \circ x_*) \\ &= (|\psi| \circ |\varphi|)(x,z); \end{aligned}$$

also, for $\varphi_i\colon X \longrightarrow\!\!\!\!\circ\; Y$ $(i \in I)$ and $x \in |X|, y \in |Y|$, one has:

$$\begin{aligned} \Big|\bigvee_i \varphi_i\Big|(x,y) &= y^* \circ \bigvee_i \varphi_i \circ x_* \\ &= \bigvee_i y^* \circ \varphi_i \circ x_* \\ &= \Big(\bigvee_i |\varphi_i|\Big)(x,y). \end{aligned}$$

Furthermore, since

$$\varphi = 1_Y^* \circ \varphi \circ 1_X^* = \bigvee_{x\in|X|, y\in|Y|} y_* \circ y^* \circ \varphi \circ x_* \circ x^* = \bigvee_{x,y} y_* \circ |\varphi|(x,y) \circ x^*,$$

φ is in fact determined by $|\varphi|$, so that $|{-}|\colon S\text{-}\mathbf{Mod} \longrightarrow V\text{-}\mathbf{Mod}$ is faithful. In order to show that $|{-}|$ is also full, given a V-module $\phi\colon |X| \longrightarrow |Y|$, one defines

$$\varphi := \bigvee_{x,y} y_* \circ \phi(x,y) \circ x^*$$

and obtains

$$\begin{aligned} |\varphi|(x',y') &= \bigvee_{x,y} (y')^* \circ y_* \circ \phi(x,y) \circ x^* \circ (x')_* \\ &= \bigvee_{x,y} |Y|(y,y') \circ \phi(x,y) \circ |X|(x',x) \\ &= \phi(x,y), \end{aligned}$$

with the last equality arising from the V-modularity of ϕ. □

Remark 3.2. The 2-functor $|{-}|\colon S\text{-}\mathbf{Mod} \longrightarrow V\text{-}\mathbf{Mod}$ of 3.1 is not only full and faitful at the 1-cell level, but also at the 2-cell level, that is:

$$|{-}|\colon S\text{-}\mathbf{Mod}(X,Y) \longrightarrow V\text{-}\mathbf{Mod}(|X|,|Y|)$$

is an order-isomorphism, for all objects X, Y in $\mathscr{K}$. Indeed, if $|\varphi| \le |\psi|$ for $\varphi, \psi\colon X \longrightarrow\!\!\!\!\circ\; Y$, then

$$\varphi = \bigvee_{x,y} y_* \circ |\varphi|(x,y) \circ x^* \le \bigvee_{x,y} y_* \circ |\psi|(x,y) \circ x^* = \psi.$$

Remark 3.3. For $\mathscr{K} = V\text{-}\mathbf{Cat}$ with a commutative unital quantale V and $S = P_V^\circ = P$ as in Example 2.4, the construction of Theorem 3.1 reproduces the given V as $P\text{-}\mathbf{Mod}(E,E) = V\text{-}\mathbf{Cat}(E,PE)$, with E the singleton V-category that is neutral w.r.t. the tensor product of $V\text{-}\mathbf{Cat}$. Indeed, V-functors $E \longrightarrow (E^{\mathrm{op}} \multimap V) \cong V$ correspond to elements of V, which produces an isomorphism $V\text{-}\mathbf{Mod}(E,E) \cong V$ of quantales. In fact, one also has (in the notation of 3.1) $|X| \cong X$ for every V-category X, in particular $|V| \cong V$ (as V-categories). Consequently, both horizontal functors in the diagram of 3.1 become equivalences of categories.

Theorem 3.1 offers two ways of making a category $\mathscr{K}$ which carries a modular monad into an ordered category, by either declaring $|-|\colon \mathscr{K} \longrightarrow V\text{-}\mathbf{Cat}$ to be full and faithful on 2-cells, or $(-)^*\colon \mathscr{K}^{\mathrm{op}} \longrightarrow S\text{-}\mathbf{Mod}$; fortunately, the two options are equivalent.

Proposition 3.4. *With the assumptions and notations of Theorem* 3.1*, one has* $|f| \leq |g|$ *in* $V\text{-}\mathbf{Cat}$ *if, and only if,* $f^* \leq g^*$ *in* $S\text{-}\mathbf{Mod}$*, for all morphisms* $f,g\colon X \longrightarrow Y$ *in* $\mathscr{K}$*.*

Proof. By the definition of the 2-categorical structure of $V\text{-}\mathbf{Cat}$, $|f| \leq |g|$ means

$$1_E^* \leq |Y|(|f|(x),|g|(x)) = x^* \circ g^* \circ f_* \circ x_*$$

for all $x \in |X|$. Consequently,

$$1_X^* = \bigvee_{x\in|X|} x_* \circ x^* \leq \bigvee_{x\in|X|} x_* \circ x^* \circ g^* \circ f_* \circ x_* \circ x^* \leq g^* \circ f_*$$

and, hence, $f^* \leq g^*$ by adjunction. The converse implication is obvious. □

Corollary 3.5. *A category with a modular monad becomes a* 2*-category when one puts*

$$f \leq g :\iff |f| \leq |g| \iff f^* \leq g^*.$$

This way all functors of the diagram of 3.1 *become full and faithful on* 2*-cells.*

4. The Hausdorff monad

For a commutative unital quantale V, let $(H, \{-\}, \bigcup)$ denote the Hausdorff monad on $V\text{-}\mathbf{Cat}$ which is a lifting of the power set monad 2.2 of $\mathbf{Set}$ along the forgetful functor $V\text{-}\mathbf{Cat} \longrightarrow \mathbf{Set}$ (see [1]). Hence, $HX = PX$ as sets, and

$$\begin{aligned} HX(A,B) &= \bigwedge_{x\in A} \bigvee_{y\in B} X(x,y) \\ &= \bigwedge_{x\in X} X(x,A) \multimap X(x,B), \end{aligned}$$

where $X(x,B) = \bigvee_{y\in B} X(x,y)$, for all $A,B \subseteq X$, X in $V\text{-}\mathbf{Cat}$. An H-module $\varphi\colon X \longrightarrow\!\!\!\!\!\circ\; Y$ is a V-functor $\varphi\colon Y \longrightarrow HX$, i.e., φ must satisfy

$$Y(y,y') \leq HX(\varphi(y),\varphi(y'))$$

for all $y,y' \in Y$, that is

$$Y(y,y') \otimes X(x,\varphi(y)) \leq X(x,\varphi(y'))$$

for all $x \in X$. Composition of φ with $\psi\colon Y \longrightarrow\!\!\!\!\!\circ\;\, Z$ is given by

$$(\psi \circ \varphi)(z) = \bigcup_{y \in \psi(z)} \varphi(y)$$

for all $z \in Z$. Finally, for $f\colon X \longrightarrow Y$ in V-**Cat**, $f^*\colon Y \longrightarrow\!\!\!\!\!\circ\;\, X$ is given by $f^*(x) = \{f(x)\}$ for all $x \in X$.

We discuss two options for making H-**Mod** into a quantaloid:

$$\begin{array}{lrlll} \text{A.} & \varphi \le \varphi' & :\iff \forall y \in Y: & \varphi(y) \subseteq \varphi'(y), & \\ \text{B.} & \varphi \lesssim \varphi' & :\iff \forall y \in Y: & \varphi(y) \le \varphi'(y) & \text{(in the } V\text{-category } HX), \\ & & \iff \forall y \in Y: & k \le HX(\varphi(y), \varphi'(y)), & \end{array}$$

for $\varphi, \varphi'\colon X \longrightarrow\!\!\!\!\!\circ\;\, Y$ in H-**Mod**.

Of course, "$\lesssim$" fails to be separated in general, but this is not essential, i.e., Definition 2.1 may be relaxed by dropping the antisymmetry requirement for the lattice structure of the hom-sets, without any detrimental effect on the subsequent theory; however, *see Remark* 4.3 *below*. With this proviso we may state:

Proposition 4.1. H-**Mod** *becomes a quantaloid under both orders,* $\le$ *and* $\lesssim$.

Proof. For $\varphi_i\colon X \longrightarrow\!\!\!\!\!\circ\;\, Y$ $(i \in I)$ in H-**Mod**, $\varphi(y) := \bigcup_{i \in I} \varphi_i(y)$ defines a supremum under either order, which is easily seen to be preserved by composition in H-**Mod** from both sides. □

The principal difference of the two structures is exhibited when we look at the order induced on V-**Cat** by $(-)^*\colon V\text{-}\mathbf{Cat} \longrightarrow H\text{-}\mathbf{Mod}$ (see 3.4): for $f, f'\colon X \longrightarrow Y$ in V-**Cat** one has:

$$\begin{array}{llllll} \text{A.} & f^* \le g^* & \iff \forall x \in X: & \{f(x)\} \subseteq \{g(x)\} & \iff & f = g, \\ \text{B.} & f^* \lesssim g^* & \iff \forall x \in X: & k \le X(f(x), g(x)) & \iff & f \le g \quad (\text{in } V\text{-}\mathbf{Cat}). \end{array}$$

Briefly, $(-)^*$ induces the discrete order on V-**Cat** under option A, and the "natural" order under option B.

Assume now that $f^*\colon Y \longrightarrow\!\!\!\!\!\circ\;\, X$ has a left adjoint $f_*\colon X \longrightarrow\!\!\!\!\!\circ\;\, Y$ in H-**Mod**. Since

$$(f^* \circ f_*)(x) = f_*(f(x)), \qquad (f_* \circ f^*)(y) = f(f_*(y))$$

for all $x \in X$, $y \in Y$, under option A the adjointness conditions amount to

$$x \in f_*(f(x)) \qquad \text{and} \qquad f_*(y) \subseteq f^{-1}(y),$$

for all $x \in X$, $y \in Y$. In addition, V-functoriality of $f_*\colon Y \longrightarrow HX$ implies

$$Y(y, y') \le \bigvee_{x' \in f_*(y')} X(x, x') \tag{$*$}$$

for all $y \in Y$, $x \in f_*(y)$. In case $V = 2$, so that $V\text{-}\mathbf{Cat} = \mathbf{Ord}$ is the category of (pre)ordered sets and monotone maps, these conditions force $f\colon X \longrightarrow Y$ to have an up-closed image: whenever $f(x) \le y'$, then $y' = f(x')$ for some $x' \ge x$. But monotonicity of f does not guarantee its image to be up-closed.

Under option B the adjointness conditions are equivalently described by

$$k \leq X\big(x, f_*(f(x))\big) \qquad \text{and} \qquad f_*(y) \subseteq \{x \in X \mid f(x) \leq y\}$$

for all $x \in X, y \in Y$. These conditions are trivially satisfied if, conversely, we now define f_* by

$$f_*(y) := \{x \in X \mid f(x) \leq y\} = \big\{x \in X \;\big|\; k \leq X(f(x), y)\big\}$$

for all $y \in Y$. In case $V = 2$, f_* satisfies also the (quite restrictive) V-functoriality condition $(*)$. In addition, condition 3 of 2.1 is trivially satisfied.

These findings may be expressed in terms of the ordinary relational composition $\circ$, as follows:

Proposition 4.2. *The Hausdorff monad H on* **Ord** *(= 2-* **Cat***) becomes modular if one orders H-* **Mod**(X, Y) *by*

$$\varphi \lesssim \varphi' \iff \varphi \subseteq \varphi' \circ (\leq_X),$$

but not when one uses ($\varphi \leq \varphi' \iff \varphi \subseteq \varphi'$). Here, a relation φ from X to Y is an H-module if, and only if, $(\leq_Y \circ \varphi) \subseteq (\varphi \circ \leq_X)$. With E a singleton set, the functor

$$|{-}|\colon H\text{-}\mathbf{Mod} \longrightarrow 2\text{-}\mathbf{Mod}$$

of 3.1 *assigns to φ the relation $\varphi \circ (\leq_X)$; it makes the (pre)ordered sets H-* **Mod**(X, Y) *and* 2-**Mod**(X, Y) *equivalent (as categories), but not necessarily isomorphic.*

Here is the reason for this last statement:

Remark 4.3. Since $\lesssim$ fails to be antisymmetric, in general the functor $|{-}|\colon H\text{-}\mathbf{Mod} \longrightarrow 2\text{-}\mathbf{Mod}$ of 4.2 is *not* necessarily faithful (on 1-cells) but satisfies only

$$|\varphi| = |\varphi'| \iff \varphi \lesssim \varphi \ \&\ \varphi' \lesssim \varphi'.$$

However, $|{-}|\colon \mathbf{Ord} \longrightarrow \mathbf{Ord} = 2\text{-}\mathbf{Cat}$ of 3.1 is an equivalence of categories.

5. A functorial correspondence between modular monads and quantales

We describe the assigment $S \longmapsto V$ of Theorem 3.1 as a functor

$$\Delta\colon \mathbf{MODMON} \longrightarrow \mathbf{Quant}.$$

Here **Quant** has as objects unital quantales, and a morphism $\Phi\colon V \longrightarrow W$ must preserve suprema and the monoid structure given by the tensor product. The (very large) category **MODMON** has as objects categories $\mathscr{K}$ equipped with a monad $S = (S, \varepsilon, \nu)$, a distinguished object E in $\mathscr{K}$ and a fixed order that makes S-**Mod** a quantaloid and S modular. A morphism

$$(F, \alpha)\colon (\mathscr{K}, (S, \varepsilon, \nu), E) \longrightarrow (\mathscr{L}, (T, \eta, \mu), D)$$

of modular monads consists of a functor $F\colon \mathscr{K} \longrightarrow \mathscr{L}$ and a natural transformation $\alpha\colon FS \longrightarrow TF$ such that $FE \cong D$, $\alpha{\cdot}F\varepsilon = \eta F$, $\alpha{\cdot}F\nu = \mu F{\cdot}T\alpha{\cdot}\alpha S$, and the induced functor

$$\widetilde{(F, \alpha)}\colon S\text{-}\mathbf{Mod} \longrightarrow T\text{-}\mathbf{Mod}$$

preserves suprema, i.e., is a morphism of quantaloids. Functoriality of

$$\widetilde{(F,\alpha)}\colon (X \xrightarrow{\varphi} Y) \longmapsto (FX \xrightarrow{\alpha_X \cdot F\varphi} FY)$$

is in fact guaranted by the preceding conditions, while preservation of suprema amounts to the condition

$$\alpha_X \cdot F\Big(\bigvee_i \varphi_i\Big) = \bigvee_i (\alpha_X \cdot F\varphi_i),$$

for all $\varphi_i\colon X \longrightarrow Y$, $i \in I$. Now $\Delta(F,\alpha)$ is simply a hom-map of the functor $\widetilde{(F,\alpha)}$:

$$\Delta(F,\alpha) := \widetilde{(F,\alpha)}_{E,E}\colon V = S\text{-}\mathbf{Mod}(E,E) \longrightarrow W = T\text{-}\mathbf{Mod}(D,D),$$

which is indeed a morphism in **Quant**. Functoriality of Δ follows from the easily checked fact

$$\widetilde{(G,\beta)(F,\alpha)} = \widetilde{(G,\beta)}\widetilde{(F,\alpha)},$$

with $(G,\beta)\colon (\mathscr{L},T,D) \longrightarrow (\mathscr{M},U,C)$ and

$$(G,\beta)(F,\alpha) = (GF, \beta F \cdot G\alpha)$$

in **MODMON**.

Calling a modular monad S on $\mathscr{K}$ *commutative* if the quantale $\Delta(\mathscr{K},S,E) = S\text{-}\mathbf{Mod}(E,E)$ is commutative, one has the restricted functor

$$\Delta\colon \mathbf{CMODMON} \longrightarrow \mathbf{CQuant}$$

of commutative objects on both sides. Next we will show that Example 2.4 provides the object function of a functor Γ in the opposite direction:

$$V \longmapsto \Gamma V = (V\text{-}\mathbf{Cat}, P_V^{\circ}, E_V),$$

with $E_V = E$ as in Remark 3.3. For a morphism $\Phi\colon V \longrightarrow W$ of commutative unital quantales, one defines the morphism $\Gamma\Phi$ in **CMODMON**, as follows. First of all, without change of notation we can regard Φ as a functor

$$V\text{-}\mathbf{Cat} \longrightarrow W\text{-}\mathbf{Cat}, \qquad X \longmapsto \Phi X = X,$$

keeping underlying sets fixed and mapping structures by Φ:

$$(\Phi X)(x,y) = \Phi(X(x,y)),$$

for all $x,y \in X$. In fact, Φ may be (more generally) regarded as a functor

$$V\text{-}\mathbf{Mod} \longrightarrow W\text{-}\mathbf{Mod}, \qquad (X \xrightarrow{\varphi} Y) \longmapsto (\Phi X \xrightarrow{\Phi\varphi} \Phi Y),$$

with $(\Phi\varphi)(x,y) = \Phi(\varphi(x,y))$ for all $x \in X$, $y \in Y$. One then has

$$\Phi(f^*) = (\Phi f)^*, \qquad \Phi(f_*) = (\Phi f)_*$$

for all $f\colon X \longrightarrow Y$ in V-**Cat**. We will show that the functor V-**Mod** $\longrightarrow$ W-**Mod** above is in fact induced by a morphism

$$(\Phi,\pi)\colon (V\text{-}\mathbf{Cat}, P^\circ_V, E_V) \longrightarrow (W\text{-}\mathbf{Cat}, P^\circ_W, E_W),$$

as

$$\widetilde{(\Phi,\pi)}\colon V\text{-}\mathbf{Mod} = P^\circ_V\text{-}\mathbf{Mod} \longrightarrow W\text{-}\mathbf{Mod} = P^\circ_W\text{-}\mathbf{Mod},$$

where now Φ is regarded as a functor V-**Cat** $\longrightarrow$ W-**Cat**. In order to define the natural transformation

$$\pi = \pi^\Phi\colon \Phi P^\circ_V \longrightarrow P^\circ_W\Phi,$$

for a V-category X we let

$$\pi_X\colon \Phi(X^{\mathrm{op}} \multimap V) \longrightarrow ((\Phi X)^{\mathrm{op}} \multimap W)$$

assign to every V-functor $\alpha\colon X^{\mathrm{op}} \longrightarrow V$ the map $(x \longmapsto \Phi(\alpha(x))$. This map is indeed a W-functor since

$$(\Phi X)(x,y) = \Phi(X(x,y)) \le \Phi(\alpha(y) \multimap \alpha(x)) \le \Phi(\alpha(y)) \multimap \Phi(\alpha(x)),$$

for all $x,y \in X$. Moreover, π_X is a W-functor since

$$\Phi\Big(\bigwedge_{x\in X} \alpha(x) \multimap \beta(x)\Big) \le \bigwedge_{x\in X} \Phi(\alpha(x) \multimap \beta(x)) \le \bigwedge_{x\in X} \Phi(\alpha(x)) \multimap \Phi(\beta(x)),$$

for all $\alpha,\beta \in P^\circ_V X$. While one immediately sees that $\widetilde{(\Phi,\pi)}$ is indeed the functor V-**Mod** $\longrightarrow$ W-**Mod** described above, it is a bit more laborious to verify the remaining requirements for (Φ,π) being a morphism in **MODMON**, namely:

Lemma 5.1. *$\pi\colon \Phi P^\circ_V \longrightarrow P^\circ_W\Phi$ is a natural transformation with $\pi\cdot\Phi\mathfrak{y}^V = \mathfrak{y}^W\Phi$ and $\pi\cdot\Phi\mathfrak{m}^V = \mathfrak{m}^W\Phi\cdot P^\circ_W\pi\cdot\pi P^\circ_V$.*

Proof. The commutativity of the diagrams

$$\begin{array}{ccc} \Phi P^\circ_V X & \xrightarrow{\Phi P^\circ_V f = P^\circ_V f} & \Phi P^\circ_V Y \\ {\scriptstyle \pi_X}\downarrow & & \downarrow{\scriptstyle \pi_Y} \\ P^\circ_W \Phi X & \xrightarrow[P^\circ_W \Phi f = P^\circ_W f]{} & P^\circ_W \Phi Y \end{array} \qquad \begin{array}{ccc} & & \Phi P^\circ_V X \\ & \nearrow^{\Phi\mathfrak{y}_X = \mathfrak{y}_X} & \downarrow{\scriptstyle \pi_X} \\ \Phi X & \searrow_{\mathfrak{y}_{\Phi X}} & P^\circ_W X \end{array}$$

is immediate, for all $f\colon X \longrightarrow Y$ in V-**Cat**:

$$\begin{aligned} P^\circ_W f(\pi_X(\alpha))(y) &= \bigvee_{x\in X} \Phi\big(Y(y, f(x))\big) \otimes \Phi(\alpha(x)) \\ &= \Phi\Big(\bigvee_{x\in X} Y(y, f(x)) \otimes \alpha(x)\Big) \\ &= \Phi(P^\circ_V f(\alpha)(y)) \\ &= \pi_Y(P^\circ_V f(\alpha))(y), \\ \pi_X(\mathfrak{y}_X(x))(x') &= \Phi(X(x', x)) = \mathfrak{y}_{\Phi X}(x)(x'), \end{aligned}$$

for all $x, x' \in X$, $y \in Y$, $\alpha \in P^\circ_V X$. For the commutativity of

$$\begin{array}{ccccc}
\Phi P^\circ_V P^\circ_V X & \xrightarrow{\pi_{P^\circ_V X}} & P^\circ_W \Phi P^\circ_V X & \xrightarrow{P^\circ_W \pi_X} & P^\circ_W P^\circ_W \Phi X \\
{\scriptstyle \Phi\mathfrak{m}_X = \mathfrak{m}_X}\downarrow & & & & \downarrow{\scriptstyle \mathfrak{m}_{\Phi X}} \\
\Phi P^\circ_V X & & \xrightarrow[\pi_X]{\qquad\qquad\qquad} & & P^\circ_W \Phi X
\end{array}$$

let $\Sigma \in P^\circ_V P^\circ_V X$ and $x \in X$; then

$$\begin{aligned}
\pi_X(\mathfrak{m}_X(\Sigma)) &= \Phi(\mathfrak{m}_X(\Sigma)(x)) \\
&= \Phi\Big(\bigvee_{\alpha \in P^\circ_V X} \Sigma(\alpha) \otimes \alpha(x)\Big) \\
&= \bigvee_{\alpha \in P^\circ_V X} \Phi(\Sigma(\alpha)) \otimes \Phi(\alpha(x)).
\end{aligned}$$

On the other hand,

$$\begin{aligned}
\mathfrak{m}_{\Phi X}\Big(P^\circ_W \pi_X(\pi_{P^\circ_V X}(\Sigma))\Big) &= \bigvee_{\beta \in P^\circ_W \Phi X} P^\circ_W \pi_X(\pi_{P^\circ_V X}(\Sigma))(\beta) \otimes \beta(x) \\
&= \bigvee_{\beta \in P^\circ_W(\Phi X)} \bigvee_{\alpha \in P^\circ_V X} P^\circ_W(\Phi X)(\beta, \pi_X(\alpha)) \otimes \pi_{P^\circ_V X}(\Sigma)(\alpha) \otimes \beta(x) \\
&= \bigvee_{\alpha \in P^\circ_V X} \Phi(\Sigma(\alpha)) \otimes \Big(\bigvee_{\beta \in P^\circ_W(\Phi X)} P^\circ_W(\Phi X)(\beta, \pi_X(\alpha)) \otimes \beta(x)\Big).
\end{aligned}$$

Hence, it suffices to show

$$\bigvee_{\beta \in P^\circ_W(\Phi X)} P^\circ_W(\Phi X)(\beta, \pi_X(\alpha)) \otimes \beta(x) = \Phi(\alpha(x));$$

but

$$P^\circ_W(\Phi X)(\beta, \pi_X(\alpha)) \otimes \beta(x) \le \Big(\beta(x) \multimap \Phi(\alpha(x))\Big) \otimes \beta(x) \le \Phi(\alpha(x))$$

for all $\beta \in P^\circ_W(\Phi X)$, and for $\beta := \Phi X(-, x)$ one has with the Yoneda Lemma

$$P^\circ_W(\Phi X)(\beta, \pi_X(\alpha)) \otimes \beta(x) \ge \pi_X(\alpha)(x) \otimes \Phi X(x, x) \ge \Phi(\alpha(x)).$$

□

Since functoriality of Δ follows immediately from the definitions, we are now ready to summarize what we have proved so far:

Theorem 5.2. *There are functors*

$$\mathbf{CQuant} \underset{\Delta}{\overset{\Gamma}{\rightleftarrows}} \mathbf{CMODMON}$$

with $\Delta\Gamma \cong 1$.

The question to "which extent" this pair of functors is adjoint is discussed in the next section.

6. An adjunction between modular monads and quantales

For a commutative monad S on $\mathscr{K}$ with distiquished object E we first revisit the functors $|{-}|\colon \mathscr{K} \longrightarrow V\text{-}\mathbf{Cat}$ and $|{-}|\colon S\text{-}\mathbf{Mod} \longrightarrow V\text{-}\mathbf{Mod}$ of 3.1 (with $V = S\text{-}\mathbf{Mod}(E,E)$) and show:

Proposition 6.1. *There is a natural transformation γ such that*

$$(|{-}|,\gamma)\colon (\mathscr{K}, S, E) \longrightarrow \Gamma V$$

is a morphism of monads with $\widetilde{(|{-}|,\gamma)} = |{-}|\colon S\text{-}\mathbf{Mod} \longrightarrow V\text{-}\mathbf{Mod}$.

Proof. For X in $\mathscr{K}$ one defines

$$\gamma_X\colon |SX| \longrightarrow P_V^{\circ}|X| = (|X|^{\mathrm{op}} \multimap V)$$

by $\gamma_X(\varphi)(x) = \varphi \circ x_*$, for all $\varphi \in |SX| = \mathscr{K}(E, SX) = S\text{-}\mathbf{Mod}(X,E)$ and $x \in |X| = \mathscr{K}(E,X)$. With $y \in |X|$ one has

$$\gamma_X(\varphi)(y) \circ |X|(x,y) = \varphi \circ y_* \circ y^* \circ x_* \leq \varphi \circ x_* = \gamma_X(\varphi)(x)$$

and then $|X|(x,y) \leq \gamma_X(\varphi)(y) \multimap \gamma_X(\varphi)(x)$, so that $\gamma_X(\varphi)\colon |X|^{\mathrm{op}} \longrightarrow V$ is indeed a V-functor. In order to show that γ_X is a V-functor, we recall from Section 2 that the V-module φ may be written as $\varphi = \varphi^* \circ \iota_X$ and, with $\psi \in |SX|$, we obtain:

$$\begin{aligned}
|SX|(\varphi,\psi) \circ \gamma_X(\varphi)(x) &= \psi^* \circ \varphi_* \circ \varphi \circ x_* \\
&= \psi^* \circ \varphi_* \circ \varphi^* \circ \iota_X \circ x_* \\
&\leq \psi^* \circ \iota_X \circ x_* \\
&= \gamma_X(\psi(x))
\end{aligned}$$

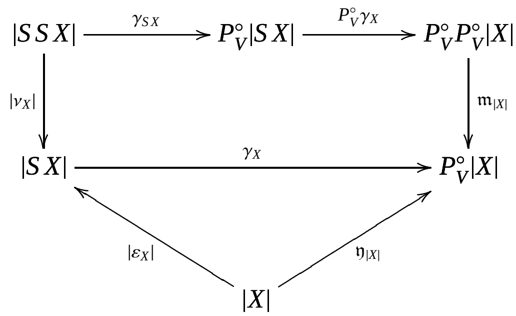
for all $x \in |X|$, hence $|SX|(\varphi,\psi) \leq P_V^{\circ}|X|(\gamma_X(\varphi), \gamma_X(\psi))$. For $f\colon X \longrightarrow Y$ in $\mathscr{K}$, the diagram

$$\begin{array}{ccc}
|SX| & \xrightarrow{\;|Sf|\;} & |SY| \\
\Big\downarrow{\scriptstyle \gamma_X} & & \Big\downarrow{\scriptstyle \gamma_Y} \\
P_V^{\circ}|X| & \xrightarrow[\;P_V^{\circ}|f|\;]{} & P_V^{\circ}|Y|
\end{array}$$

commutes since for all $\varphi \in |SX|$ and $y \in |Y|$ one has:

$$\begin{aligned}
P_V^{\circ}|f|(\gamma_X(\varphi))(y) &= \bigvee_{x \in |X|} \gamma_X(\varphi)(x) \circ |Y|(y, |f|(x)) \\
&= \bigvee_{x \in |X|} \varphi^* \circ \iota_X \circ x_* \circ (f \cdot x)^* \circ y_* \\
&= \varphi^* \circ \iota_X \circ \Big(\bigvee_{x \in |X|} x_* \circ x^*\Big) \circ f^* \circ y_* \\
&= \varphi^* \circ \iota_X \circ f^* \circ y_* \\
&= \varphi^* \circ (Sf)^* \circ \iota_Y \circ y_* \\
&= (Sf \cdot \varphi)^* \circ \iota_Y \circ y_* \\
&= \gamma_Y(Sf \cdot \varphi)(y) \\
&= (\gamma_Y \cdot |Sf|)(\varphi)(y).
\end{aligned}$$

Hence, γ is a natural transformation. Finally, we must check that the following diagram commutes:



But for all $x, y \in |X|$ one has:

$$\gamma_X(|\varepsilon_X|(x))(y) = (\varepsilon_X \cdot x)^* \circ \iota_X \circ y_* = x^* \circ \varepsilon_X^* \circ \iota_X \circ y_* = x^* \circ y_* = \mathfrak{y}_{|X|}(x)(y).$$

Furthermore, for all $\theta \in |S\,S\,X|$ and $x \in X$ one obtains:

$$\begin{aligned}\gamma_X(|\nu_X|(\theta))(x) &= \gamma_X(\nu_X \cdot \theta)(x)\\ &= \theta^* \circ \nu_X^* \circ \iota_X \circ x_*\\ &= \theta^* \circ \iota_{SX} \circ \iota_X \circ x_*\\ &= \bigvee_{\varphi \in |S\,X|} \theta^* \circ \iota_{SX} \circ \varphi_* \circ \varphi^* \circ \iota_X \circ x_*.\end{aligned}$$

Using the Yoneda Lemma one may rewrite

$$\varphi^* \circ \iota_X \circ x_* = \gamma_X(\varphi)(x) = \bigvee_{\alpha \in P_V^\circ|X|} P_V^\circ|X|(\alpha, \gamma_X(\varphi)) \circ \alpha(x),$$

which then gives

$$\begin{aligned}\gamma_X(|\nu_X|(\theta))(x) &= \bigvee_{\alpha \in P_V^\circ|X|} \bigvee_{\varphi \in S\,X} \theta^* \circ \iota_{SX} \circ \varphi_* \circ P_V^\circ|X|(\alpha, \gamma_X(\varphi)) \circ \alpha(x)\\ &= \bigvee_{\alpha \in P_V^\circ|X|} P_V^\circ \gamma_X(\gamma_{SX}(\theta))(\alpha) \circ \alpha(x)\\ &= \mathfrak{m}_{|X|}\Big(P_V^\circ \gamma_X(\gamma_{SX}(\theta))\Big)(x).\end{aligned}$$

□

For a commutative modular monad S on $\mathscr{K}$ with distiguished object E and a commutative unital quantale W, from 5.2 and 6.1 one obtains the map

$$\begin{aligned}\mathbf{CQuant}(\Delta(\mathscr{K}, S, E), W) &\longrightarrow \mathbf{CMODMON}((\mathscr{K}, S, E), \Gamma W)\\ \Phi &\longmapsto \Gamma\Phi \cdot (|{-}|, \gamma)\end{aligned}$$

which, however, cannot be expected to be surjective, not even "up to isomorphism": for a morphism $(F, \alpha)\colon (\mathscr{K}, S, E) \longrightarrow (W\text{-}\mathbf{Cat}, P_W^\circ, E_W)$ to be in its image, FX must have underlying set $\mathscr{K}(E, X)$, but one should allow for appropriate isomorphisms. Hence we must restrict the codomain of the above map appropriately.

Definition 6.2. (1) *A morphism* $(F,\alpha)\colon (\mathscr{K},S,E) \longrightarrow (\mathscr{L},T,D)$ *of modular monads is* ***representable*** *if there is a natural isomorphism* $\tau\colon \mathscr{K}(E,-) \longrightarrow \mathscr{L}(D,F-)$.

(2) *A* 2-***cell*** $\theta\colon (F,\alpha) \longrightarrow (G,\beta)$ *of morphisms* $(F,\alpha),(G,\beta)\colon (\mathscr{K},S,E) \longrightarrow (\mathscr{L},T,D)$ *of modular monads is a natural transformation* $\theta\colon F \longrightarrow G$ *with* $T\theta\cdot\alpha = \beta\cdot\theta S$ *;* (F,α) *and* (G,β) *are* ***isomorphic*** *if* θ *can be chosen to be an isomorphism, i.e., if all* θ_X *are isomorphisms.*

Remarks 6.3. (1) By the Yoneda Lemma, a natural transformation $\tau\colon \mathscr{K}(E,-) \longrightarrow \mathscr{L}(D,F-)$ is completely determined by a morphism $i\colon D \longrightarrow FE$ in $\mathscr{L}$, as $\tau_X(x) = Fx\cdot i$ for all $x \in \mathscr{K}(E,X)$. Since $\mathscr{K}(E,E) = \{1_E\}$ and $FE \cong D$ one sees that (F,α) *is representable if, and only if, the maps* $F_{E,X}\colon \mathscr{K}(E,X) \longrightarrow \mathscr{L}(FE,FX)$ *are bijective for all objects* X *in* $\mathscr{K}$.

(2) A 2-cell $\theta\colon (F,\alpha) \longrightarrow (G,\beta) : (\mathscr{K},S,E) \longrightarrow (\mathscr{L},T,D)$ induces a natural transformation $\theta^*\colon \widetilde{(G,\beta)} \longrightarrow \widetilde{(F,\alpha)}\colon S\text{-}\mathbf{Mod} \longrightarrow T\text{-}\mathbf{Mod}$ with $(\theta^*)_X = (\theta_X)^*$ for all object X in $\mathscr{K}$. Since $\mathscr{L}(D,D) \cong \mathscr{L}(D,FE) \cong \mathscr{L}(D,GE)$ are singleton sets, $\theta_E = j\cdot i^{-1}$ is an isomorphism, with the unique morphisms $i\colon D \longrightarrow FE$, $j\colon D \longrightarrow GE$ in $\mathscr{L}$. Consequently, for every $v \in V = S\text{-}\mathbf{Mod}(E,E)$ one has a commutative diagram

$$\begin{array}{ccc} FE & \xrightarrow{\;\widetilde{(F,\alpha)}(v)\;} & FE \\ {\scriptstyle\theta_E^*}\uparrow & & \uparrow{\scriptstyle\theta_E^*} \\ GE & \xrightarrow[\;\widetilde{(G,\alpha)}(v)\;]{} & GE \end{array}$$

with the horizontal arrows therefore determining the same element in $W = T\text{-}\mathbf{Mod}(D,D)$. Consequently: *if there is a* 2*-cell* $(F,\alpha) \longrightarrow (G,\beta)$ *of morphisms of modular monads, then* $\Delta(F,\alpha) = \Delta(G,\beta)$.

(3) For a 2-cell θ as above, from the naturality condition $\theta_X\cdot Fx = Gx\cdot\theta_E$ for all $x \in \mathscr{K}(E,X)$ and the fact that θ_E is the only morphism in $\mathscr{L}(FE,GE)$ one derives immediately: *if the family* $(Fx)_{x\in|X|}$ *is jointly epic, then there is at most one* 2*-cell* $(F,\alpha) \longrightarrow (G,\beta)$. When (F,α) is representable and $(\mathscr{L},T,D) = \Gamma W = (W\text{-}\mathbf{Cat}, P^\circ_W, E_W)$ for a commutative unitale quantale W, the epi-condition is certainly satisfied since there is a bijection

$$F_{E,X}\colon \mathscr{K}(E,X) \longrightarrow W\text{-}\mathbf{Cat}(FE,FX) \cong FX.$$

(4) For any morphism $(F,\alpha)\colon (\mathscr{K},S,E) \longrightarrow (\mathscr{L},T,D)$ of modular monads one has

$$\widetilde{(F,\alpha)}(f^*) = (Ff)^* \qquad \text{and} \qquad \widetilde{(F,\alpha)}(f_*) = (Ff)_*,$$

for all morphisms $f\colon X \longrightarrow Y$ in $\mathscr{K}$. Indeed,

$$\widetilde{(F,\alpha)}(f^*) = \alpha_Y\cdot F\varepsilon_Y\cdot Ff = \eta_{FY}\cdot Ff = (Ff)^*;$$

and since both, $\widetilde{(F,\alpha)}(f_*)$ and $(Ff)_*$ are left adjoint to $(Ff)^*$, also the second claim holds.

We can now set up the category **RCMODMON** whose objects are commutative modular monads (as in **CMODMON**) but whose morphisms are isomorphism classes of representable morphisms $(F,\alpha)\colon (\mathscr{K},S,E) \longrightarrow (\mathscr{L},T,D)$; we denote the class of (F,α) by $[F,\alpha]$. By Remark 6.3(2), the functor

$$\Delta\colon\ \mathbf{RCMODMON} \longrightarrow \mathbf{CQuant}, \qquad [F,\alpha] \longmapsto \Delta(F,\alpha),$$

is well defined, and we can now state:

Theorem 6.4. *Δ has a full and faithful right adjoint Γ. Hence,* **CQuant** *is equivalent to a full reflective subcategory of* **RCMODMON**.

Proof. With Γ defined by $\Phi \longmapsto [\Phi, \pi^{\Phi}]$ (see Section 5) we must prove that every morphism

$$[F,\alpha]\colon (\mathscr{K}, S, E) \longrightarrow \Gamma W = (W\text{-}\mathbf{Cat}, P^{\circ}_{W}, E_W)$$

in **RCMODMON** with a commutative unital quantale W factors as $[F,\alpha] = \Gamma\Phi\cdot[|{-}|,\gamma]$, with a uniquely determined morphism $\Phi\colon V = \Delta(\mathscr{K}, S, E) \longrightarrow W$ of quantales. By Remark 6.3(2), such Φ must necessarily satisfy

$$(\widetilde{\Phi,\pi^{\phi}})(\widetilde{|{-}|,\gamma}) \cong (\widetilde{F,\alpha});$$

in particular, the diagram

$$\begin{array}{ccc} V = S\text{-}\mathbf{Mod}(E,E) & \xrightarrow{\ \sim\ } & V\text{-}\mathbf{Mod}(|E|,|E|) \\ & \searrow^{(\widetilde{F,\alpha})_{E,E}} & \downarrow \Phi \\ & & W\text{-}\mathbf{Mod}(E_W,E_W) \cong W \end{array}$$

must commute. In other words, up to trivial isomorphisms, Φ is neccessarily given by $(\widetilde{F,\alpha})_{E,E}$.

Conversely, for proving its existence, let us define $\Phi\colon V \longrightarrow W$ by $\Phi(v) = (\widetilde{F,\alpha})_{E,E}(v)$ for all $v \in V$ (thus ignoring trivial bijections). Then Φ is certainly a morphism of quantales (see Section 5). Furthermore, denoting the underlying **Set**-functors of $V\text{-}\mathbf{Cat}$ and by U_V and U_W, respectively, from the representability of (F,α) we obtain a natural isomorphism

$$U_W F \xrightarrow{\ \sim\ } \mathscr{K}(E,-) = U_V|{-}| = U_W\Phi|{-}|,$$

with $|{-}|\colon \mathscr{K} \longrightarrow V\text{-}\mathbf{Cat}$ and $\Phi\colon V\text{-}\mathbf{Cat} \longrightarrow W\text{-}\mathbf{Cat}$ as in 3.1 and Section 5. We must now lift this **Set**-based isomorphism to a $W\text{-}\mathbf{Cat}$-based isomorphism $\theta\colon F \xrightarrow{\ \sim\ } \Phi|{-}|$. For ease of computation, we may without loss of generality assume that the **Set**-based isomorphism is actually an identity; hence, FX has underlying set $\mathscr{K}(E,X)$, for all objects X in $\mathscr{K}$, and we must show that FX and $\Phi|X|$ have the same W-category structure. But for all $x, y \in |X| = \mathscr{K}(E,X)$ one has with 6.3(4):

$$\begin{aligned} (\Phi|X|)(x,y) &= \Phi(|X|(x,y)) \\ &= \Phi(y^{*} \circ x_{*}) \\ &= (\widetilde{F,\alpha})(y^{*}) \circ (\widetilde{F,\alpha})(x_{*}) \\ &= (Fy)^{*} \circ (Fx)_{*} \\ &= (FX)(x,y). \end{aligned}$$

Finally, in order to confirm θ as a 2-cell $(F,\alpha) \xrightarrow{\ \sim\ } \Gamma\Phi\cdot(|{-}|,\gamma)$, under the assumption $\theta = 1$ we must show that the diagram

$$\begin{array}{ccccc} \Phi|SX| & \xrightarrow{\ \Phi_{\gamma_X} = \gamma_X\ } & \Phi(P^{\circ}_{V}|X|) & \xrightarrow{\ \pi_X\ } & P^{\circ}_{W}\Phi X \\ \| & & & & \| \\ FSX & & \xrightarrow[\ \alpha_X\ ]{} & & P^{\circ}_{W}FX \end{array}$$

commutes, for all objects X. To this end, let us first observe that α_X may be considered a W-module $FX \longrightarrow\!\!\!\!\circ\, FSX$, and as such is represented as

$$\alpha_X = (\widetilde{F,\alpha})(\iota_X),$$

with $\iota_X\colon X \longrightarrow\!\!\!\!\circ\, SX$ in S-**Mod** (see Section 2). Now, for all $\varphi \in |SX|$ and $x \in X$, we obtain:

$$\begin{aligned}
\pi_X(\gamma_X(\varphi))(x) &= \Phi(\gamma_X(\varphi)(x)) \\
&= \Phi(\varphi \circ x_*) \\
&= \Phi(\varphi^* \circ \iota_X \circ x_*) \\
&= (\widetilde{F,\alpha})(\varphi^* \circ \iota_X \circ x_*) \\
&= (\widetilde{F,\alpha})(\varphi^*) \circ (\widetilde{F,\alpha})(\iota_X) \circ (\widetilde{F,\alpha})(x_*) \\
&= (F\varphi)^* \circ \alpha_X \circ (Fx)_* \\
&= \alpha_X(\varphi)(x).
\end{aligned}$$

□

Remark 6.5. While **MODMON** carries the structure of a 2-category (see 6.2(2)), the full extent of this structure is of limited interest for our purposes, since Δ maps every 2-cell to an identity morphism: see 6.3(2). Likewise, the natural 2-categorical structure of **CQuant**, given by ordering its hom-sets pointwise, is not helpful for our purposes: if $\Phi, \Psi\colon V \longrightarrow W$ are morphisms of commutative quantales with $\Phi(v) \le \Psi(v)$ for all $v \in V$, the natural transformation $\theta\colon \Phi \longrightarrow \Psi\colon V$-**Cat** $\longrightarrow W$-**Cat** with $\theta_X = 1_X$ (at the **Set**-level) will in general *not* give a 2-cell $\Gamma\Phi \longrightarrow \Gamma\Psi$.

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