

Kleisli enriched

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Abstract

For a monad S on a category \mathcal{K} whose Kleisli category is a quantaloid, we introduce the notion of modularity, in such a way that morphisms in the Kleisli category may be regarded as V -(bi)modules (= profunctors, distributors), for some quantale V . The assignment $S \mapsto V$ is shown to belong to a global adjunction which, in the opposite direction, associates with every (commutative, unital) quantale V the prototypical example of a modular monad, namely the presheaf monad on $V\text{-Cat}$, the category of (small) V -categories. We discuss in particular the question whether the Hausdorff monad on $V\text{-Cat}$ is modular.

Keywords: modular monad, Kleisli category, quantale, quantaloid, V -(bi)module, V -category, power-set monad, presheaf monad, Hausdorff monad.

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1. Introduction

For a monad S on a category \mathcal{K} , a morphism in the Kleisli category of S is given by a morphism of type $Y \longrightarrow SX$ in \mathcal{K} . As the carrier of the free Eilenberg-Moore algebra over X , naturally SX carries additional structure which may be inherited by the relevant hom-set of the Kleisli category. For example, when S is the power-set monad on \mathbf{Set} , so that $SX = PX$ is the free sup-lattice over the set X , the Kleisli category is the (dual of the) category of sets and relations and, hence, a quantaloid, i.e., a **Sup**-enriched category. Less trivially, and more generally, taking presheaves over a (small) category X defines a monad P on \mathbf{Cat} whose Kleisli category is the (bi)category of categories and bimodules (= profunctors, distributors), the rich structure of which is a fundamental tool for a substantial body of categorical research. We refer the reader in particular to [12], [6], [5] and [4], and the extensive lists of references in these papers which point the reader also to the origins of a theme that seems to have interested researchers for some forty years. Some of these papers consider the presheaf monad in the enriched context (see [8]), i.e., for $V\text{-Cat}$, where V is a symmetric monoidal-closed category (rather than the classical $V = \mathbf{Set}$), or even a bicategory (see [10], for example). The more manageable case when the bicategory is just a quantaloid has been considered by Stubbe [13] who exhibits the passage from V to the category of V -bimodules as a morphism of quantaloids.

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In this paper we consider the further simplified case when V is a quantale, i.e., a one-object quantaloid which, based on Lawvere’s treatment of metric spaces [9] and Barr’s presentation of topological spaces [2], has been used to set up a common syntax for various categories of interest in analysis and topology; see, for example, [3], [11], [7], [14]. Specifically, with (the dual of) the Kleisli category of the presheaf monad P on $V\text{-Cat}$ describing precisely the quantaloid $V\text{-Mod}$ of V -categories and V -modules, we ask ourselves the question when the morphisms of the Kleisli category of an arbitrary monad S on a category may be treated as V -modules, for some quantale V . To this end we introduce the notion of a *modular monad* on an abstract category \mathcal{K} which asks its Kleisli category to be a quantaloid (and, hence, a 2-category) in which \mathcal{K} -morphisms have adjoints. By means of a distinguished “unital” object E in \mathcal{K} one may then associate with the monad S a quantale V and establish a fully faithful “comparison functor” from the (dual of the) Kleisli category to $V\text{-Mod}$. With some natural restrictions on both the objects and morphisms, this fully faithful functor plays the role of a unit of an adjunction between a (very large) category of modular monads and the category of (commutative unital) quantales, the counits of which are isomorphisms. In other words, assigning to every V the presheaf monad on $V\text{-Cat}$ defines, up to categorical equivalence, a full reflective embedding of the category of quantales into a category of modular monads. In setting up this category, some care must be given to the definition of its morphisms since the assignments $S \longmapsto V$ and $V \longmapsto P$ behave surprisingly crudely with respect to the natural 2-categorical structures of these categories. We have therefore used 2-cells only to the minimal extent necessary to answer our original question.

The paper is written in a largely self-contained style; it therefore recalls some known facts, giving sufficient details in particular on our prototypical example of a modular monad, the presheaf monad on $V\text{-Cat}$ (Section 2). Having already set up a “modular terminology” for the dual of the Kleisli category of a modular monad in that introductory section, in Section 3 we prove the comparison theorem with the modules of actual V -categories, for some suitable V . Section 4 contains a detailed discussion of the question to which extent the Hausdorff monad (see [1]) is modular. Finally, global correspondences between quantales and modular monads are established in Sections 5 and 6, first in terms of functors that are partially (pseudo-)inverse to each other, and then in terms of the somewhat surprising adjunction that exhibits the ordinary category of quantales as a very substantial part of a very large environment of (certain) modular monads.

2. Modular monads

Let $S = (S, \varepsilon, \nu)$ be a monad on a category \mathcal{K} . We denote the opposite of the Kleisli category of S by

$$S\text{-Mod}.$$

Hence, its objects are the objects of \mathcal{K} , and a morphism $\varphi: X \multimap Y$, also called *S -module* from X to Y , is given by a \mathcal{K} -morphism $\varphi: Y \longrightarrow SX$; composition with $\psi: Y \multimap Z$ is defined by

$$\psi \circ \varphi := \nu_X \cdot S\varphi \cdot \psi,$$

and $1_X^* := \varepsilon_X: X \multimap X$ is the identity morphism on X in $S\text{-Mod}$. Extending this notation, one has the right-adjoint functor

$$(-)^*: \mathcal{K}^{\text{op}} \longrightarrow S\text{-Mod}$$

which sends $f: X \longrightarrow Y$ in \mathcal{K} to $f^* := \varepsilon_Y \cdot f: Y \rightrightarrows X$ in $S\text{-Mod}$. We denote its left adjoint by

$$\widehat{(-)}: S\text{-Mod} \longrightarrow \mathcal{K}^{\text{op}},$$

sending X to SX and $\varphi: X \rightrightarrows Y$ to $\hat{\varphi} := \nu_X \cdot S\varphi: SY \longrightarrow SX$ in \mathcal{K} . The adjunction produces two factorizations of φ , namely

$$\varphi = \hat{\varphi} \cdot \varepsilon_Y \quad \text{in } \mathcal{K} \quad \text{and} \quad \varphi = \varphi^* \circ \iota_X \quad \text{in } S\text{-Mod},$$

with the morphisms ε_X serving as counits (in \mathcal{K}^{op}), and the morphisms $\iota_X := 1_{SX}: X \rightrightarrows SX$ as units (in $S\text{-Mod}$). Units and counits are connected by the triangular equalities (which are special cases of the factorization)

$$\widehat{\iota_X} \cdot \varepsilon_{SX} = 1_{SX} \quad \text{in } \mathcal{K} \quad \text{and} \quad \varepsilon_X^* \circ \iota_X = 1_X^* \quad \text{in } S\text{-Mod}.$$

Definition 2.1. We call the monad S on \mathcal{K} *modular* if

1. $S\text{-Mod}$ carries the structure of a quantaloid, that is: every hom-set carries the structure of a complete lattice, such that composition in $S\text{-Mod}$ from either side preserves arbitrary suprema;
2. for every morphism $f: X \longrightarrow Y$ in \mathcal{K} , $f^*: Y \rightrightarrows X$ has a left adjoint in the 2-category $S\text{-Mod}$, that is: there exists $f_*: X \rightrightarrows Y$ in $S\text{-Mod}$ with $1_X^* \leq f^* \circ f_*$ and $f_* \circ f^* \leq 1_Y^*$;
3. there is an object E in \mathcal{K} with $\mathcal{K}(E, E) = \{1_E\}$ and

$$\bigvee_{x: E \rightarrow X} x_* \circ x^* = 1_X^*$$

for all X in \mathcal{K} .

For a modular monad S , we always fix the order that makes $S\text{-Mod}$ a quantaloid and assume a fixed choice of the left adjoints f_* and the distinguished object E ; in other words, modularity is not considered as a property of the monad, but as a structure on it.

Example 2.2. For the power-set monad $(P, \{-, \cup\})$ on \mathbf{Set} , when one considers maps $\varphi: Y \longrightarrow PX$ as relations $\varphi: X \rightrightarrows Y$ (by writing $x \varphi y$ instead of $x \in \varphi(y)$), $P\text{-Mod}$ is simply the category \mathbf{Rel} of sets and relations. Its hom-sets inherit the inclusion order of power-sets, which makes $P\text{-Mod}$ a quantaloid. One takes $f_*: Y \longrightarrow PX$ to be the inverse-image function of $f: X \longrightarrow Y$ in \mathbf{Set} , and with E a singleton set, condition 3 of 2.1 amounts to the trivial statement

$$(\exists x \in X : u = x \& x = v) \iff u = v$$

for all $u, v \in X$. Hence, P is modular.

Example 2.3. Replacing the 2-element chain in $PX = 2^X$ by an arbitrary frame V , one generalizes the previous example, as follows. For $f: X \longrightarrow Y$ in \mathbf{Set} , $P_V f: P_V X = V^X \longrightarrow P_V Y$ be left adjoint to $V^f: V^Y \longrightarrow V^X, \beta \mapsto \beta \cdot f$; hence

$$(P_V f)(\alpha)(y) = \bigvee_{x \in f^{-1}(y)} \alpha(x),$$

for all $\alpha: X \longrightarrow V, y \in Y$. The maps

$$x \xrightarrow{\delta_x} P_V X \quad \text{and} \quad P_V P_V X \xrightarrow{v_X} P_V X$$

with $\delta_X(x)(x') = \top$ (the top element in V) if $x = x'$ and \perp (bottom) else, and with

$$v_X(\Sigma)(x) = \bigvee_{\alpha \in P_V X} \Sigma(\alpha) \wedge \alpha(x)$$

for all $\Sigma: V^X \longrightarrow V, x, x' \in X$, give P_V the structure of a monad. Using P instead of P_V , we can describe $P\text{-Mod}$ equivalently as the quantaloid $V\text{-Rel}$ of sets with V -valued relations $\varphi: X \longrightarrow Y$ as morphisms. Indeed, maps $\varphi: Y \longrightarrow PX = V^X$ correspond bijectively to maps $\tilde{\varphi}: X \times Y \longrightarrow V$, and composition in $P\text{-Mod}$ becomes the ordinary composition of V -valued relations:

$$\widetilde{\psi \circ \varphi}(x, z) = (\tilde{\psi} \circ \tilde{\varphi})(x, z) = \bigvee_{y \in Y} \tilde{\psi}(y, z) \wedge \tilde{\varphi}(x, y),$$

for $\psi: Y \longrightarrow Z, x \in X, z \in Z$. The left adjoint f_* of f^* for $f: X \longrightarrow Y$ is obtained by interchanging variables:

$$f_*(y)(x) = f^*(x)(y) = \top$$

if $f(x) = y$, and \perp else. Condition 3 of 2.1 is, as in the case $V = 2$, trivially satisfied also in general. Consequently, $P = P_V$ is a modular monad on \mathbf{Set} .

Example 2.4. Let V be a unital quantale (= one-object quantaloid), i.e. a complete lattice with a binary associative operation \otimes and a neutral element k such that \otimes preserves suprema in each variable. (Every frame V as in 2.3 is a quantale, with $\otimes = \wedge, k = \top$.) The category $V\text{-Cat}$ of (small) V -categories and V -functors has as objects sets X which come with a function $X \times X \longrightarrow V$ (whose value on (x, y) we denote by $X(x, y)$) such that

$$k \leq X(x, x) \quad \text{and} \quad X(y, z) \otimes X(x, y) \leq X(x, z);$$

morphisms $f: X \longrightarrow Y$ satisfy $X(x, y) \leq Y(f(x), f(y))$, for all $x, y, z \in V$. The quantale V itself is a V -category, with the V -category structure $V(v, w) = v \multimap w$ given by its own ‘‘internal hom’’ defined by

$$z \leq v \multimap w \iff z \otimes v \leq w$$

for all $z, v, w \in V$. Moreover, $V\text{-Cat}$ has an ‘‘internal hom’’ with

$$X \multimap Y = V\text{-Cat}(X, Y) \quad \text{and} \\ (X \multimap Y)(f, g) = \bigvee_{x \in X} Y(f(x), g(x)).$$

When V is commutative, $V\text{-Cat}$ is symmetric monoidal closed, with $X \otimes Y = X \times Y$ and

$$(X \otimes Y)((x, y), (x', y')) = X(x, x') \otimes Y(y, y'),$$

and one can also form the opposite X^{op} of a V -category X , with $X^{\text{op}}(x, y) = X(y, x)$. Now, the Yoneda embedding

$$\eta_X: X \longrightarrow P_V^\circ X := (X^{\text{op}} \multimap V), \quad x \longmapsto X(-, x),$$

provides the unit of the presheaf monad (P_V°, η, m) of $V\text{-Cat}$, as follows. Writing P instead of P_V° , for $f: X \longrightarrow Y$, the V -functor $Pf: PX \longrightarrow PY$ is defined by

$$(Pf)(\alpha)(y) = \bigvee_{x \in X} Y(y, f(x)) \otimes \alpha(x)$$

for all $\alpha \in PX$, $y \in Y$, and the monad multiplication $m_X: PPX \longrightarrow PX$ is given by

$$m_X(\Sigma)(x) = \bigvee_{\alpha \in PX} \Sigma(\alpha) \otimes \alpha(x).$$

We claim that *the category $P\text{-Mod}$ is precisely the category $V\text{-Mod}$ whose objects are V -categories, and whose morphisms $\varphi: X \longrightarrow Y$ are V -(bi)modules, also called V -distributors or V -profunctors, given by functions $\varphi: X \times Y \longrightarrow V$ satisfying*

$$Y(y_1, y_2) \otimes \varphi(x_2, y_1) \otimes X(x_1, x_2) \leq \varphi(x_1, y_2) \quad (*)$$

for all $x_1, x_2 \in X$, $y_1, y_2 \in Y$; composition with $\psi: Y \longrightarrow Z$ is defined by

$$(\psi \circ \varphi)(x, z) = \bigvee_{y \in Y} \psi(y, z) \otimes \varphi(x, y). \quad (**)$$

PROOF. Since $(*)$ is equivalent to $\varphi: X^{\text{op}} \otimes Y \longrightarrow V$ being a V -functor we may as well think of φ as a V -functor $Y \longrightarrow PX$, writing $\varphi(x, y)$ as $\varphi(y)(x)$. Hence, all we need to verify is that the composition $(**)$ in $V\text{-Mod}$ coincides with the Kleisli composition of $P\text{-Mod}$, i.e.,

$$\bigvee_{y \in Y} \varphi(y)(x) \otimes \psi(z)(y) = (m_X \cdot P\varphi \cdot \psi)(z)(x) \quad (***)$$

for all $\varphi: Y \longrightarrow PX$, $\psi: Z \longrightarrow PY$ in $V\text{-Cat}$, $x \in X$, $z \in Z$. By the Yoneda Lemma the left-hand side of $(***)$ may be rewritten and compared to the right-hand side, as follows:

$$\begin{aligned} \bigvee_{y \in Y} PX(\eta_X(x), \varphi(y)) \otimes \psi(z)(y) &= (P\varphi)(\psi(z))(\eta_X(x)) \\ &\leq (P\varphi)(\psi(z))(\eta_X(x)) \otimes \eta_X(x)(x) \\ &\leq \bigvee_{\alpha \in PX} (P\varphi)(\psi(z))(\alpha) \otimes \alpha(x) \\ &= m_X((P\varphi)(\psi(z)))(x). \end{aligned}$$

For “ \geq ”, since

$$\left(\bigvee_{x' \in X} \alpha(x') \multimap \varphi(y)(x') \right) \otimes \alpha(x) \leq (\alpha(x) \multimap \varphi(y)(x)) \otimes \alpha(x) \leq \varphi(y)(x),$$

one has, for all $\alpha \in PX$,

$$\begin{aligned} (P\varphi)(\psi(z))(\alpha) \otimes \alpha(x) &= \bigvee_{y \in Y} PX(\alpha, \varphi(y)) \otimes \psi(z)(y) \otimes \alpha(x) \\ &\leq \bigvee_{y \in Y} \varphi(y)(x) \otimes \psi(z)(y), \end{aligned}$$

as desired. □

With suprema formed pointwise (as in $V\text{-Rel}$ of 2.3), $V\text{-Mod}$ becomes a quantaloid, and it is easy to check the remaining conditions to confirm that P is modular, by putting $f_*(y)(x) := Y(f(x), y)$ for all $x \in X, y \in Y, f: X \longrightarrow Y$ in $V\text{-Cat}$.

Note that Example 2.3 is a special case of 2.4, by restriction to *discrete* V -categories, i.e., sets.

In Section 4 we discuss an example showing that conditions 2, 3 of 2.1 do not follow from condition 1 of 2.1.

3. Exhibiting Kleisli morphisms as V -modules

The following theorem shows that Example 2.4 exhibits the prototypical modular monad, and it justifies the module terminology for the morphisms of its Kleisli category.

Theorem 3.1. *Let $S = (S, \varepsilon, \nu)$ be a modular monad on a category \mathcal{K} . Then there is a unital quantale V and a functor $|-|: \mathcal{K} \longrightarrow V\text{-Cat}$ that can be lifted to a full and faithful homomorphism $|-|: S\text{-Mod} \longrightarrow V\text{-Mod}$ of quantaloids such that*

$$\begin{array}{ccc}
 S\text{-Mod} & \xrightarrow{|-|} & V\text{-Mod} \\
 \uparrow (-)^* & & \uparrow (-)^* \\
 \mathcal{K}^{\text{op}} & \xrightarrow{|-|^{\text{op}}} & (V\text{-Cat})^{\text{op}}
 \end{array}$$

commutes.

PROOF. The monoid structure of $V := S\text{-Mod}(E, E) = \mathcal{K}(E, SE)$ makes V a unital quantale since $S\text{-Mod}$ is a quantaloid. The hom-functor $|-| = \mathcal{K}(E, -)$ takes values in $V\text{-Cat}$ if, for a \mathcal{K} object X , we put

$$|X|(x, y) := y^* \circ x_*$$

for all $x, y \in |X|$. Indeed, the adjunctions $x_* \dashv x^*$ show the V -category laws for $|X|$, and for a morphism $f: X \longrightarrow Y$ in \mathcal{K} one obtains the V -functoriality of $|f|$ from

$$|X|(x, y) = y^* \circ 1_X^* \circ x_* \leq y^* \circ f^* \circ f_* \circ x_* \leq (f \cdot y)^* \circ (f \cdot x)_* = |Y|(|f|(x), |f|(x))$$

for all $x, y \in |X|$; here we used the fact that the local adjunctions $f_* \dashv f^*$ compose, so that functoriality of $(-)^*$ produces a pseudofunctor $(-)_*: \mathcal{K} \longrightarrow S\text{-Mod}$. For $\varphi: X \longrightarrow Y$ in $S\text{-Mod}$ one defines $|\varphi|: |X| \longrightarrow |Y|$ in $V\text{-Mod}$ by

$$|\varphi|(x, y) = y^* \circ \varphi \circ x_*$$

for $x \in |X|, y \in |Y|$; V -modularity follows immediately from the local adjunctions. Also, for $f: X \longrightarrow Y$ in \mathcal{K} , one has

$$|f^*|(y, x) = x^* \circ f^* \circ y_* = (f \cdot x)^* \circ y_* = |Y|(y, |f|(x)) = |f|^*(y, x),$$

for all $x \in |X|$, $y \in |Y|$. Next we show that $|-|: S\text{-}\mathbf{Mod} \longrightarrow V\text{-}\mathbf{Mod}$ preserves composition and suprema. In fact, for $\psi: Y \longrightarrow Z$ and $x \in |X|$, $z \in |Z|$ one has:

$$\begin{aligned} |\psi \circ \varphi|(x, z) &= z^* \circ \psi \circ 1_Y^* \circ \varphi \circ x_* \\ &= z^* \circ \psi \circ \left(\bigvee_{y \in |Y|} y_* \circ y^* \right) \circ \varphi \circ x_* \\ &= \bigvee_{y \in |Y|} (z^* \circ \psi \circ y_*) \circ (y^* \circ \varphi \circ x_*) \\ &= (|\psi| \circ |\varphi|)(x, z); \end{aligned}$$

also, for $\varphi_i: X \longrightarrow Y$ ($i \in I$) and $x \in |X|$, $y \in |Y|$, one has:

$$\begin{aligned} \left| \bigvee_i \varphi_i \right|(x, y) &= y^* \circ \bigvee_i \varphi_i \circ x_* \\ &= \bigvee_i y^* \circ \varphi_i \circ x_* \\ &= \left(\bigvee_i |\varphi_i| \right)(x, y). \end{aligned}$$

Furthermore, since

$$\varphi = 1_Y^* \circ \varphi \circ 1_X^* = \bigvee_{x \in |X|, y \in |Y|} y_* \circ y^* \circ \varphi \circ x_* \circ x^* = \bigvee_{x, y} y_* \circ |\varphi|(x, y) \circ x^*,$$

φ is in fact determined by $|\varphi|$, so that $|-|: S\text{-}\mathbf{Mod} \longrightarrow V\text{-}\mathbf{Mod}$ is faithful. In order to show that $|-|$ is also full, given a V -module $\phi: |X| \longrightarrow |Y|$, one defines

$$\varphi := \bigvee_{x, y} y_* \circ \phi(x, y) \circ x^*$$

and obtains

$$\begin{aligned} |\varphi|(x', y') &= \bigvee_{x, y} (y')^* \circ y_* \circ \phi(x, y) \circ x^* \circ (x')_* \\ &= \bigvee_{x, y} |Y|(y, y') \circ \phi(x, y) \circ |X|(x', x) \\ &= \phi(x, y), \end{aligned}$$

with the last equality arising from the V -modularity of ϕ . □

Remark 3.2. The 2-functor $|-|: S\text{-}\mathbf{Mod} \longrightarrow V\text{-}\mathbf{Mod}$ of 3.1 is not only full and faithful at the 1-cell level, but also at the 2-cell level, that is:

$$|-|: S\text{-}\mathbf{Mod}(X, Y) \longrightarrow V\text{-}\mathbf{Mod}(|X|, |Y|)$$

is an order-isomorphism, for all objects X, Y in \mathcal{X} . Indeed, if $|\varphi| \leq |\psi|$ for $\varphi, \psi: X \longrightarrow Y$, then

$$\varphi = \bigvee_{x, y} y_* \circ |\varphi|(x, y) \circ x^* \leq \bigvee_{x, y} y_* \circ |\psi|(x, y) \circ x^* = \psi.$$

Remark 3.3. For $\mathcal{K} = V\text{-Cat}$ with a commutative unital quantale V and $S = P_V^\circ = P$ as in Example 2.4, the construction of Theorem 3.1 reproduces the given V as $P\text{-Mod}(E, E) = V\text{-Cat}(E, PE)$, with E the singleton V -category that is neutral w.r.t. the tensor product of $V\text{-Cat}$. Indeed, V -functors $E \longrightarrow (E^{\text{op}} \multimap V) \cong V$ correspond to elements of V , which produces an isomorphism $V\text{-Mod}(E, E) \cong V$ of quantales. In fact, one also has (in the notation of 3.1) $|X| \cong X$ for every V -category X , in particular $|V| \cong V$ (as V -categories). Consequently, both horizontal functors in the diagram of 3.1 become equivalences of categories.

Theorem 3.1 offers two ways of making a category \mathcal{K} which carries a modular monad into an ordered category, by either declaring $|-|: \mathcal{K} \longrightarrow V\text{-Cat}$ to be full and faithful on 2-cells, or $(-)^*: \mathcal{K}^{\text{op}} \longrightarrow S\text{-Mod}$; fortunately, the two options are equivalent.

Proposition 3.4. *With the assumptions and notations of Theorem 3.1, one has $|f| \leq |g|$ in $V\text{-Cat}$ if, and only if, $f^* \leq g^*$ in $S\text{-Mod}$, for all morphisms $f, g: X \longrightarrow Y$ in \mathcal{K} .*

PROOF. By the definition of the 2-categorical structure of $V\text{-Cat}$, $|f| \leq |g|$ means

$$1_E^* \leq |Y|(|f|(x), |g|(x)) = x^* \circ g^* \circ f_* \circ x_*$$

for all $x \in |X|$. Consequently,

$$1_X^* = \bigvee_{x \in |X|} x_* \circ x^* \leq \bigvee_{x \in |X|} x_* \circ x^* \circ g^* \circ f_* \circ x_* \circ x^* \leq g^* \circ f_*$$

and, hence, $f^* \leq g^*$ by adjunction. The converse implication is obvious. \square

Corollary 3.5. *A category with a modular monad becomes a 2-category when one puts*

$$f \leq g : \iff |f| \leq |g| \iff f^* \leq g^*.$$

This way all functors of the diagram of 3.1 become full and faithful on 2-cells.

4. The Hausdorff monad

For a commutative unital quantale V , let $(H, \{-, \cup\})$ denote the Hausdorff monad on $V\text{-Cat}$ which is a lifting of the power set monad 2.2 of \mathbf{Set} along the forgetful functor $V\text{-Cat} \longrightarrow \mathbf{Set}$ (see [1]). Hence, $HX = PX$ as sets, and

$$\begin{aligned} HX(A, B) &= \bigwedge_{x \in A} \bigvee_{y \in B} X(x, y) \\ &= \bigwedge_{x \in X} X(x, A) \multimap X(x, B), \end{aligned}$$

where $X(x, B) = \bigvee_{y \in B} X(x, y)$, for all $A, B \subseteq X$, X in $V\text{-Cat}$. An H -module $\varphi: X \longrightarrow Y$ is a V -functor $\varphi: Y \longrightarrow HX$, i.e., φ must satisfy

$$Y(y, y') \leq HX(\varphi(y), \varphi(y'))$$

for all $y, y' \in Y$, that is

$$Y(y, y') \otimes X(x, \varphi(y)) \leq X(x, \varphi(y'))$$

for all $x \in X$. Composition of φ with $\psi: Y \longrightarrow Z$ is given by

$$(\psi \circ \varphi)(z) = \bigcup_{y \in \psi(z)} \varphi(y)$$

for all $z \in Z$. Finally, for $f: X \longrightarrow Y$ in $V\text{-Cat}$, $f^*: Y \longrightarrow X$ is given by $f^*(x) = \{f(x)\}$ for all $x \in X$.

We discuss two options for making $H\text{-Mod}$ into a quantaloid:

- A. $\varphi \leq \varphi' : \iff \forall y \in Y : \quad \varphi(y) \subseteq \varphi'(y),$
- B. $\varphi \lesssim \varphi' : \iff \forall y \in Y : \quad \varphi(y) \leq \varphi'(y) \quad (\text{in the } V\text{-category } HX),$
 $\iff \forall y \in Y : \quad k \leq HX(\varphi(y), \varphi'(y)),$

for $\varphi, \varphi': X \longrightarrow Y$ in $H\text{-Mod}$.

Of course, “ \lesssim ” fails to be separated in general, but this is not essential, i.e., Definition 2.1 may be relaxed by dropping the antisymmetry requirement for the lattice structure of the hom-sets, without any detrimental effect on the subsequent theory; however, *see Remark 4.3 below*. With this proviso we may state:

Proposition 4.1. *$H\text{-Mod}$ becomes a quantaloid under both orders, \leq and \lesssim .*

PROOF. For $\varphi_i: X \longrightarrow Y$ ($i \in I$) in $H\text{-Mod}$, $\varphi(y) := \bigcup_{i \in I} \varphi_i(y)$ defines a supremum under either order, which is easily seen to be preserved by composition in $H\text{-Mod}$ from both sides. \square

The principal difference of the two structures is exhibited when we look at the order induced on $V\text{-Cat}$ by $(-)^*: V\text{-Cat} \longrightarrow H\text{-Mod}$ (see 3.4): for $f, f': X \longrightarrow Y$ in $V\text{-Cat}$ one has:

- A. $f^* \leq g^* \iff \forall x \in X : \quad \{f(x)\} \subseteq \{g(x)\} \iff f = g,$
- B. $f^* \lesssim g^* \iff \forall x \in X : \quad k \leq X(f(x), g(x)) \iff f \leq g \quad (\text{in } V\text{-Cat}).$

Briefly, $(-)^*$ induces the discrete order on $V\text{-Cat}$ under option A, and the “natural” order under option B.

Assume now that $f^*: Y \longrightarrow X$ has a left adjoint $f_*: X \longrightarrow Y$ in $H\text{-Mod}$. Since

$$(f^* \circ f_*)(x) = f_*(f(x)), \quad (f_* \circ f^*)(y) = f(f_*(y))$$

for all $x \in X, y \in Y$, under option A the adjointness conditions amount to

$$x \in f_*(f(x)) \quad \text{and} \quad f_*(y) \subseteq f^{-1}(y),$$

for all $x \in X, y \in Y$. In addition, V -functoriality of $f_*: Y \longrightarrow HX$ implies

$$Y(y, y') \leq \bigvee_{x' \in f_*(y')} X(x, x') \quad (*)$$

for all $y \in Y, x \in f_*(y)$. In case $V = 2$, so that $V\text{-Cat} = \mathbf{Ord}$ is the category of (pre)ordered sets and monotone maps, these conditions force $f: X \longrightarrow Y$ to have an up-closed image: whenever $f(x) \leq y'$, then $y' = f(x')$ for some $x' \geq x$. But monotonicity of f does not guarantee its image to be up-closed.

Under option B the adjointness conditions are equivalently described by

$$k \leq X(x, f_*(f(x))) \quad \text{and} \quad f_*(y) \subseteq \{x \in X \mid f(x) \leq y\}$$

for all $x \in X, y \in Y$. These conditions are trivially satisfied if, conversely, we now define f_* by

$$f_*(y) := \{x \in X \mid f(x) \leq y\} = \{x \in X \mid k \leq X(f(x), y)\}$$

for all $y \in Y$. In case $V = 2$, f_* satisfies also the (quite restrictive) V -functoriality condition (*). In addition, condition 3 of 2.1 is trivially satisfied.

These findings may be expressed in terms of the ordinary relational composition \circ , as follows:

Proposition 4.2. *The Hausdorff monad H on \mathbf{Ord} (= 2-Cat) becomes modular if one orders $H\text{-Mod}(X, Y)$ by*

$$\varphi \lesssim \varphi' \iff \varphi \subseteq \varphi' \circ (\leq_X),$$

but not when one uses $(\varphi \leq \varphi' \iff \varphi \subseteq \varphi')$. Here, a relation φ from X to Y is an H -module if, and only if, $(\leq_Y \circ \varphi) \subseteq (\varphi \circ \leq_X)$. With E a singleton set, the functor

$$|-|: H\text{-Mod} \longrightarrow 2\text{-Mod}$$

of 3.1 assigns to φ the relation $\varphi \circ (\leq_X)$; it makes the (pre)ordered sets $H\text{-Mod}(X, Y)$ and $2\text{-Mod}(X, Y)$ equivalent (as categories), but not necessarily isomorphic.

Here is the reason for this last statement:

Remark 4.3. Since \lesssim fails to be antisymmetric, in general the functor $|-|: H\text{-Mod} \longrightarrow 2\text{-Mod}$ of 4.2 is *not* necessarily faithful (on 1-cells) but satisfies only

$$|\varphi| = |\varphi'| \iff \varphi \lesssim \varphi' \ \& \ \varphi' \lesssim \varphi.$$

However, $|-|: \mathbf{Ord} \longrightarrow \mathbf{Ord} = 2\text{-Cat}$ of 3.1 is an equivalence of categories.

5. A functorial correspondence between modular monads and quantales

We describe the assignment $S \longmapsto V$ of Theorem 3.1 as a functor

$$\Delta: \mathbf{MODMON} \longrightarrow \mathbf{Quant}.$$

Here \mathbf{Quant} has as objects unital quantales, and a morphism $\Phi: V \longrightarrow W$ must preserve suprema and the monoid structure given by the tensor product. The (very large) category \mathbf{MODMON} has as objects categories \mathcal{K} equipped with a monad $S = (S, \varepsilon, \nu)$, a distinguished object E in \mathcal{K} and a fixed order that makes $S\text{-Mod}$ a quantaloid and S modular. A morphism

$$(F, \alpha): (\mathcal{K}, (S, \varepsilon, \nu), E) \longrightarrow (\mathcal{L}, (T, \eta, \mu), D)$$

of modular monads consists of a functor $F: \mathcal{K} \longrightarrow \mathcal{L}$ and a natural transformation $\alpha: FS \longrightarrow TF$ such that $FE \cong D$, $\alpha \cdot F\varepsilon = \eta F$, $\alpha \cdot F\nu = \mu F \cdot T\alpha \cdot S$, and the induced functor

$$\widetilde{(F, \alpha)}: S\text{-Mod} \longrightarrow T\text{-Mod}$$

preserves suprema, i.e., is a morphism of quantaloids. Functoriality of

$$\widetilde{(F, \alpha)}: (X \xrightarrow{\varphi} Y) \longmapsto (FX \xrightarrow{\alpha_X \cdot F\varphi} FY)$$

is in fact guaranteed by the preceding conditions, while preservation of suprema amounts to the condition

$$\alpha_X \cdot F\left(\bigvee_i \varphi_i\right) = \bigvee_i (\alpha_X \cdot F\varphi_i),$$

for all $\varphi_i: X \longrightarrow Y$, $i \in I$. Now $\Delta(F, \alpha)$ is simply a hom-map of the functor $\widetilde{(F, \alpha)}$:

$$\Delta(F, \alpha) := \widetilde{(F, \alpha)}_{E,E}: V = S\text{-}\mathbf{Mod}(E, E) \longrightarrow W = T\text{-}\mathbf{Mod}(D, D),$$

which is indeed a morphism in **Quant**. Functoriality of Δ follows from the easily checked fact

$$(G, \beta)\widetilde{(F, \alpha)} = \widetilde{(G, \beta)}\widetilde{(F, \alpha)},$$

with $(G, \beta): (\mathcal{L}, T, D) \longrightarrow (\mathcal{M}, U, C)$ and

$$(G, \beta)(F, \alpha) = (GF, \beta F \cdot G\alpha)$$

in **MODMON**.

Calling a modular monad S on \mathcal{H} *commutative* if the quantale $\Delta(\mathcal{H}, S, E) = S\text{-}\mathbf{Mod}(E, E)$ is commutative, one has the restricted functor

$$\Delta: \mathbf{CMODMON} \longrightarrow \mathbf{CQuant}$$

of commutative objects on both sides. Next we will show that Example 2.4 provides the object function of a functor Γ in the opposite direction:

$$V \longmapsto \Gamma V = (V\text{-}\mathbf{Cat}, P_V^0, E_V),$$

with $E_V = E$ as in Remark 3.3. For a morphism $\Phi: V \longrightarrow W$ of commutative unital quantales, one defines the morphism $\Gamma\Phi$ in **CMODMON**, as follows. First of all, without change of notation we can regard Φ as a functor

$$V\text{-}\mathbf{Cat} \longrightarrow W\text{-}\mathbf{Cat}, \quad X \longmapsto \Phi X = X,$$

keeping underlying sets fixed and mapping structures by Φ :

$$(\Phi X)(x, y) = \Phi(X(x, y)),$$

for all $x, y \in X$. In fact, Φ may be (more generally) regarded as a functor

$$V\text{-}\mathbf{Mod} \longrightarrow W\text{-}\mathbf{Mod}, \quad (X \xrightarrow{\varphi} Y) \longmapsto (\Phi X \xrightarrow{\Phi\varphi} \Phi Y),$$

with $(\Phi\varphi)(x, y) = \Phi(\varphi(x, y))$ for all $x \in X, y \in Y$. One then has

$$\Phi(f^*) = (\Phi f)^*, \quad \Phi(f_*) = (\Phi f)_*$$

for all $f: X \longrightarrow Y$ in $V\text{-Cat}$. We will show that the functor $V\text{-Mod} \longrightarrow W\text{-Mod}$ above is in fact induced by a morphism

$$(\Phi, \pi): (V\text{-Cat}, P_V^\circ, E_V) \longrightarrow (W\text{-Cat}, P_W^\circ, E_W),$$

as

$$\widetilde{(\Phi, \pi)}: V\text{-Mod} = P_V^\circ\text{-Mod} \longrightarrow W\text{-Mod} = P_W^\circ\text{-Mod},$$

where now Φ is regarded as a functor $V\text{-Cat} \longrightarrow W\text{-Cat}$. In order to define the natural transformation

$$\pi = \pi^\Phi: \Phi P_V^\circ \longrightarrow P_W^\circ \Phi,$$

for a V -category X we let

$$\pi_X: \Phi(X^{\text{op}} \multimap V) \longrightarrow ((\Phi X)^{\text{op}} \multimap W)$$

assign to every V -functor $\alpha: X^{\text{op}} \longrightarrow V$ the map $(x \longmapsto \Phi(\alpha(x)))$. This map is indeed a W -functor since

$$(\Phi X)(x, y) = \Phi(X(x, y)) \leq \Phi(\alpha(y) \multimap \alpha(x)) \leq \Phi(\alpha(y)) \multimap \Phi(\alpha(x)),$$

for all $x, y \in X$. Moreover, π_X is a W -functor since

$$\Phi\left(\bigwedge_{x \in X} \alpha(x) \multimap \beta(x)\right) \leq \bigwedge_{x \in X} \Phi(\alpha(x) \multimap \beta(x)) \leq \bigwedge_{x \in X} \Phi(\alpha(x)) \multimap \Phi(\beta(x)),$$

for all $\alpha, \beta \in P_V^\circ X$. While one immediately sees that $\widetilde{(\Phi, \pi)}$ is indeed the functor $V\text{-Mod} \longrightarrow W\text{-Mod}$ described above, it is a bit more laborious to verify the remaining requirements for (Φ, π) being a morphism in **MODMON**, namely:

Lemma 5.1. $\pi: \Phi P_V^\circ \longrightarrow P_W^\circ \Phi$ is a natural transformation with $\pi \cdot \Phi \eta^V = \eta^W \Phi$ and $\pi \cdot \Phi m^V = m^W \Phi \cdot P_W^\circ \pi \cdot P_V^\circ$.

PROOF. The commutativity of the diagrams

$$\begin{array}{ccc} \Phi P_V^\circ X & \xrightarrow{\Phi P_V^\circ f = P_V^\circ f} & \Phi P_V^\circ Y \\ \pi_X \downarrow & & \downarrow \pi_Y \\ P_W^\circ \Phi X & \xrightarrow{P_W^\circ \Phi f = P_W^\circ f} & P_W^\circ \Phi Y \end{array} \quad \begin{array}{ccc} & \nearrow \Phi \eta_X = \eta_X & \Phi P_V^\circ X \\ \Phi X & & \downarrow \pi_X \\ & \searrow \eta_{\Phi X} & P_W^\circ X \end{array}$$

is immediate, for all $f: X \longrightarrow Y$ in $V\text{-Cat}$:

$$\begin{aligned} P_W^\circ f(\pi_X(\alpha))(y) &= \bigvee_{x \in X} \Phi(Y(y, f(x))) \otimes \Phi(\alpha(x)) \\ &= \Phi\left(\bigvee_{x \in X} Y(y, f(x)) \otimes \alpha(x)\right) \\ &= \Phi(P_V^\circ f(\alpha)(y)) \\ &= \pi_Y(P_V^\circ f(\alpha))(y), \\ \pi_X(\eta_X(x))(x') &= \Phi(X(x', x)) = \eta_{\Phi X}(x)(x'), \end{aligned}$$

for all $x, x' \in X, y \in Y, \alpha \in P_V^\circ X$. For the commutativity of

$$\begin{array}{ccccc}
\Phi P_V^\circ P_V^\circ X & \xrightarrow{\pi_{P_V^\circ X}} & P_W^\circ \Phi P_V^\circ X & \xrightarrow{P_W^\circ \pi_X} & P_W^\circ P_W^\circ \Phi X \\
\downarrow \Phi m_X = m_X & & & & \downarrow m_{\Phi X} \\
\Phi P_V^\circ X & \xrightarrow{\pi_X} & P_W^\circ \Phi X & &
\end{array}$$

let $\Sigma \in P_V^\circ P_V^\circ X$ and $x \in X$; then

$$\begin{aligned}
\pi_X(m_X(\Sigma)) &= \Phi(m_X(\Sigma)(x)) \\
&= \Phi\left(\bigvee_{\alpha \in P_V^\circ X} \Sigma(\alpha) \otimes \alpha(x)\right) \\
&= \bigvee_{\alpha \in P_V^\circ X} \Phi(\Sigma(\alpha)) \otimes \Phi(\alpha(x)).
\end{aligned}$$

On the other hand,

$$\begin{aligned}
m_{\Phi X}(P_W^\circ \pi_X(\pi_{P_V^\circ X}(\Sigma))) &= \bigvee_{\beta \in P_W^\circ \Phi X} P_W^\circ \pi_X(\pi_{P_V^\circ X}(\Sigma))(\beta) \otimes \beta(x) \\
&= \bigvee_{\beta \in P_W^\circ (\Phi X)} \bigvee_{\alpha \in P_V^\circ X} P_W^\circ (\Phi X)(\beta, \pi_X(\alpha)) \otimes \pi_{P_V^\circ X}(\Sigma)(\alpha) \otimes \beta(x) \\
&= \bigvee_{\alpha \in P_V^\circ X} \Phi(\Sigma(\alpha)) \otimes \left(\bigvee_{\beta \in P_W^\circ (\Phi X)} P_W^\circ (\Phi X)(\beta, \pi_X(\alpha)) \otimes \beta(x)\right).
\end{aligned}$$

Hence, it suffices to show

$$\bigvee_{\beta \in P_W^\circ (\Phi X)} P_W^\circ (\Phi X)(\beta, \pi_X(\alpha)) \otimes \beta(x) = \Phi(\alpha(x));$$

but

$$P_W^\circ (\Phi X)(\beta, \pi_X(\alpha)) \otimes \beta(x) \leq (\beta(x) \multimap \Phi(\alpha(x))) \otimes \beta(x) \leq \Phi(\alpha(x))$$

for all $\beta \in P_W^\circ (\Phi X)$, and for $\beta := \Phi X(-, x)$ one has with the Yoneda Lemma

$$P_W^\circ (\Phi X)(\beta, \pi_X(\alpha)) \otimes \beta(x) \geq \pi_X(\alpha)(x) \otimes \Phi X(x, x) \geq \Phi(\alpha(x)).$$

□

Since functoriality of Δ follows immediately from the definitions, we are now ready to summarize what we have proved so far:

Theorem 5.2. *There are functors*

$$\mathbf{CQuant} \begin{array}{c} \xrightarrow{\Gamma} \\ \xleftarrow{\Delta} \end{array} \mathbf{CMODMON}$$

with $\Delta\Gamma \cong 1$.

The question to “which extent” this pair of functors is adjoint is discussed in the next section.

6. An adjunction between modular monads and quantales

For a commutative monad S on \mathcal{K} with distinguished object E we first revisit the functors $|-|: \mathcal{K} \longrightarrow V\text{-Cat}$ and $|-|: S\text{-Mod} \longrightarrow V\text{-Mod}$ of 3.1 (with $V = S\text{-Mod}(E, E)$) and show:

Proposition 6.1. *There is a natural transformation γ such that*

$$(|-, \gamma): (\mathcal{K}, S, E) \longrightarrow \Gamma V$$

is a morphism of monads with $(\widetilde{|-|}, \gamma) = |-|: S\text{-Mod} \longrightarrow V\text{-Mod}$.

PROOF. For X in \mathcal{K} one defines

$$\gamma_X: |SX| \longrightarrow P_V^\circ |X| = (|X|^{\text{op}} \multimap V)$$

by $\gamma_X(\varphi)(x) = \varphi \circ x_*$, for all $\varphi \in |SX| = \mathcal{K}(E, SX) = S\text{-Mod}(X, E)$ and $x \in |X| = \mathcal{K}(E, X)$. With $y \in |X|$ one has

$$\gamma_X(\varphi)(y) \circ |X|(x, y) = \varphi \circ y_* \circ y^* \circ x_* \leq \varphi \circ x_* = \gamma_X(\varphi)(x)$$

and then $|X|(x, y) \leq \gamma_X(\varphi)(y) \multimap \gamma_X(\varphi)(x)$, so that $\gamma_X(\varphi): |X|^{\text{op}} \longrightarrow V$ is indeed a V -functor. In order to show that γ_X is a V -functor, we recall from Section 2 that the V -module φ may be written as $\varphi = \varphi^* \circ \iota_X$ and, with $\psi \in |SX|$, we obtain:

$$\begin{aligned} |SX|(\varphi, \psi) \circ \gamma_X(\varphi)(x) &= \psi^* \circ \varphi_* \circ \varphi \circ x_* \\ &= \psi^* \circ \varphi_* \circ \varphi^* \circ \iota_X \circ x_* \\ &\leq \psi^* \circ \iota_X \circ x_* \\ &= \gamma_X(\psi)(x) \end{aligned}$$

for all $x \in |X|$, hence $|SX|(\varphi, \psi) \leq P_V^\circ |X|(\gamma_X(\varphi), \gamma_X(\psi))$. For $f: X \longrightarrow Y$ in \mathcal{K} , the diagram

$$\begin{array}{ccc} |SX| & \xrightarrow{|Sf|} & |SY| \\ \gamma_X \downarrow & & \downarrow \gamma_Y \\ P_V^\circ |X| & \xrightarrow{P_V^\circ |f|} & P_V^\circ |Y| \end{array}$$

commutes since for all $\varphi \in |SX|$ and $y \in |Y|$ one has:

$$\begin{aligned} P_V^\circ |f|(\gamma_X(\varphi))(y) &= \bigvee_{x \in |X|} \gamma_X(\varphi)(x) \circ |Y|(y, |f|(x)) \\ &= \bigvee_{x \in |X|} \varphi^* \circ \iota_X \circ x_* \circ (f \cdot x)^* \circ y_* \\ &= \varphi^* \circ \iota_X \circ \left(\bigvee_{x \in |X|} x_* \circ x^* \right) \circ f^* \circ y_* \\ &= \varphi^* \circ \iota_X \circ f^* \circ y_* \\ &= \varphi^* \circ (Sf)^* \circ \iota_Y \circ y_* \\ &= (Sf \cdot \varphi)^* \circ \iota_Y \circ y_* \\ &= \gamma_Y(Sf \cdot \varphi)(y) \\ &= (\gamma_Y \cdot |Sf|)(\varphi)(y). \end{aligned}$$

Hence, γ is a natural transformation. Finally, we must check that the following diagram commutes:

$$\begin{array}{ccccc}
|SSX| & \xrightarrow{\gamma_{SX}} & P_V^\circ |SX| & \xrightarrow{P_V^\circ \gamma_X} & P_V^\circ P_V^\circ |X| \\
\downarrow |\nu_X| & & & & \downarrow \mathfrak{m}_{|X|} \\
|SX| & \xrightarrow{\gamma_X} & & & P_V^\circ |X| \\
& \nwarrow |\varepsilon_X| & & \nearrow \eta_{|X|} & \\
& & |X| & &
\end{array}$$

But for all $x, y \in |X|$ one has:

$$\gamma_X(|\varepsilon_X|(x))(y) = (\varepsilon_X \cdot x)^* \circ \iota_X \circ y_* = x^* \circ \varepsilon_X^* \circ \iota_X \circ y_* = x^* \circ y_* = \eta_{|X|}(x)(y).$$

Furthermore, for all $\theta \in |SSX|$ and $x \in X$ one obtains:

$$\begin{aligned}
\gamma_X(|\nu_X|(\theta))(x) &= \gamma_X(\nu_X \cdot \theta)(x) \\
&= \theta^* \circ \nu_X^* \circ \iota_X \circ x_* \\
&= \theta^* \circ \iota_{SX} \circ \iota_X \circ x_* \\
&= \bigvee_{\varphi \in |SX|} \theta^* \circ \iota_{SX} \circ \varphi_* \circ \varphi^* \circ \iota_X \circ x_*.
\end{aligned}$$

Using the Yoneda Lemma one may rewrite

$$\varphi^* \circ \iota_X \circ x_* = \gamma_X(\varphi)(x) = \bigvee_{\alpha \in P_V^\circ |X|} P_V^\circ |X|(\alpha, \gamma_X(\varphi)) \circ \alpha(x),$$

which then gives

$$\begin{aligned}
\gamma_X(|\nu_X|(\theta))(x) &= \bigvee_{\alpha \in P_V^\circ |X|} \bigvee_{\varphi \in |SX|} \theta^* \circ \iota_{SX} \circ \varphi_* \circ P_V^\circ |X|(\alpha, \gamma_X(\varphi)) \circ \alpha(x) \\
&= \bigvee_{\alpha \in P_V^\circ |X|} P_V^\circ \gamma_X(\gamma_{SX}(\theta))(\alpha) \circ \alpha(x) \\
&= \mathfrak{m}_{|X|}(P_V^\circ \gamma_X(\gamma_{SX}(\theta)))(x).
\end{aligned}$$

□

For a commutative modular monad S on \mathcal{K} with distinguished object E and a commutative unital quantale W , from 5.2 and 6.1 one obtains the map

$$\begin{aligned}
\mathbf{CQuant}(\Delta(\mathcal{K}, S, E), W) &\longrightarrow \mathbf{CMODMON}((\mathcal{K}, S, E), \Gamma W) \\
\Phi &\longmapsto \Gamma\Phi \cdot (|-|, \gamma)
\end{aligned}$$

which, however, cannot be expected to be surjective, not even “up to isomorphism”: for a morphism $(F, \alpha): (\mathcal{K}, S, E) \longrightarrow (W\text{-Cat}, P_W^\circ, E_W)$ to be in its image, FX must have underlying set $\mathcal{K}(E, X)$, but one should allow for appropriate isomorphisms. Hence we must restrict the codomain of the above map appropriately.

Definition 6.2. (1) A morphism $(F, \alpha): (\mathcal{K}, S, E) \longrightarrow (\mathcal{L}, T, D)$ of modular monads is **representable** if there is a natural isomorphism $\tau: \mathcal{K}(E, -) \longrightarrow \mathcal{L}(D, F-)$.

(2) A 2-cell $\theta: (F, \alpha) \longrightarrow (G, \beta)$ of morphisms $(F, \alpha), (G, \beta): (\mathcal{K}, S, E) \longrightarrow (\mathcal{L}, T, D)$ of modular monads is a natural transformation $\theta: F \longrightarrow G$ with $T\theta \cdot \alpha = \beta \cdot \theta S$; (F, α) and (G, β) are **isomorphic** if θ can be chosen to be an isomorphism, i.e., if all θ_X are isomorphisms.

Remarks 6.3. (1) By the Yoneda Lemma, a natural transformation $\tau: \mathcal{K}(E, -) \longrightarrow \mathcal{L}(D, F-)$ is completely determined by a morphism $i: D \longrightarrow FE$ in \mathcal{L} , as $\tau_X(x) = Fx \cdot i$ for all $x \in \mathcal{K}(E, X)$. Since $\mathcal{K}(E, E) = \{1_E\}$ and $FE \cong D$ one sees that (F, α) is representable if, and only if, the maps $F_{E,X}: \mathcal{K}(E, X) \longrightarrow \mathcal{L}(FE, FX)$ are bijective for all objects X in \mathcal{K} .

(2) A 2-cell $\theta: (F, \alpha) \longrightarrow (G, \beta): (\mathcal{K}, S, E) \longrightarrow (\mathcal{L}, T, D)$ induces a natural transformation $\theta^*: (\widetilde{G}, \beta) \longrightarrow (\widetilde{F}, \alpha): S\text{-Mod} \longrightarrow T\text{-Mod}$ with $(\theta^*)_X = (\theta_X)^*$ for all object X in \mathcal{K} . Since $\mathcal{L}(D, D) \cong \mathcal{L}(D, FE) \cong \mathcal{L}(D, GE)$ are singleton sets, $\theta_E = j \cdot i^{-1}$ is an isomorphism, with the unique morphisms $i: D \longrightarrow FE$, $j: D \longrightarrow GE$ in \mathcal{L} . Consequently, for every $v \in V = S\text{-Mod}(E, E)$ one has a commutative diagram

$$\begin{array}{ccc}
 FE & \xrightarrow{(\widetilde{F}, \alpha)(v)} & FE \\
 \uparrow \theta_E^* & & \uparrow \theta_E^* \\
 GE & \xrightarrow{(\widetilde{G}, \alpha)(v)} & GE
 \end{array}$$

with the horizontal arrows therefore determining the same element in $W = T\text{-Mod}(D, D)$. Consequently: if there is a 2-cell $(F, \alpha) \longrightarrow (G, \beta)$ of morphisms of modular monads, then $\Delta(F, \alpha) = \Delta(G, \beta)$.

(3) For a 2-cell θ as above, from the naturality condition $\theta_X \cdot Fx = Gx \cdot \theta_E$ for all $x \in \mathcal{K}(E, X)$ and the fact that θ_E is the only morphism in $\mathcal{L}(FE, GE)$ one derives immediately: if the family $(Fx)_{x \in |X|}$ is jointly epic, then there is at most one 2-cell $(F, \alpha) \longrightarrow (G, \beta)$. When (F, α) is representable and $(\mathcal{L}, T, D) = \Gamma W = (W\text{-Cat}, P_W^{\circ}, E_W)$ for a commutative unital quantale W , the epi-condition is certainly satisfied since there is a bijection

$$F_{E,X}: \mathcal{K}(E, X) \longrightarrow W\text{-Cat}(FE, FX) \cong FX.$$

(4) For any morphism $(F, \alpha): (\mathcal{K}, S, E) \longrightarrow (\mathcal{L}, T, D)$ of modular monads one has

$$(\widetilde{F}, \alpha)(f^*) = (Ff)^* \quad \text{and} \quad (\widetilde{F}, \alpha)(f_*) = (Ff)_*,$$

for all morphisms $f: X \longrightarrow Y$ in \mathcal{K} . Indeed,

$$(\widetilde{F}, \alpha)(f^*) = \alpha_Y \cdot F\varepsilon_Y \cdot Ff = \eta_{FY} \cdot Ff = (Ff)^*;$$

and since both, $(\widetilde{F}, \alpha)(f_*)$ and $(Ff)_*$ are left adjoint to $(Ff)^*$, also the second claim holds.

We can now set up the category **RCMODMON** whose objects are commutative modular monads (as in **CMODMON**) but whose morphisms are isomorphism classes of representable morphisms $(F, \alpha): (\mathcal{K}, S, E) \longrightarrow (\mathcal{L}, T, D)$; we denote the class of (F, α) by $[F, \alpha]$. By Remark 6.3(2), the functor

$$\Delta: \mathbf{RCMODMON} \longrightarrow \mathbf{CQuant}, \quad [F, \alpha] \longmapsto \Delta(F, \alpha),$$

is well defined, and we can now state:

Theorem 6.4. Δ has a full and faithful right adjoint Γ . Hence, **CQuant** is equivalent to a full reflective subcategory of **RCMODMON**.

PROOF. With Γ defined by $\Phi \longmapsto [\Phi, \pi^\Phi]$ (see Section 5) we must prove that every morphism

$$[F, \alpha]: (\mathcal{X}, S, E) \longrightarrow \Gamma W = (W\text{-Cat}, P_W^\circ, E_W)$$

in **RCMODMON** with a commutative unital quantale W factors as $[F, \alpha] = \Gamma\Phi \cdot [|\cdot|, \gamma]$, with a uniquely determined morphism $\Phi: V = \Delta(\mathcal{X}, S, E) \longrightarrow W$ of quantales. By Remark 6.3(2), such Φ must necessarily satisfy

$$(\widetilde{\Phi}, \widetilde{\pi^\Phi})(|\cdot|, \gamma) \cong (\widetilde{F}, \widetilde{\alpha});$$

in particular, the diagram

$$\begin{array}{ccc} V = S\text{-Mod}(E, E) & \xrightarrow{\sim} & V\text{-Mod}(|E|, |E|) \\ & \searrow^{(\widetilde{F}, \widetilde{\alpha})_{E,E}} & \downarrow \Phi \\ & & W\text{-Mod}(E_W, E_W) \cong W \end{array}$$

must commute. In other words, up to trivial isomorphisms, Φ is necessarily given by $(\widetilde{F}, \widetilde{\alpha})_{E,E}$.

Conversely, for proving its existence, let us define $\Phi: V \longrightarrow W$ by $\Phi(v) = (\widetilde{F}, \widetilde{\alpha})_{E,E}(v)$ for all $v \in V$ (thus ignoring trivial bijections). Then Φ is certainly a morphism of quantales (see Section 5). Furthermore, denoting the underlying **Set**-functors of $V\text{-Cat}$ and by U_V and U_W , respectively, from the representability of (F, α) we obtain a natural isomorphism

$$U_W F \xrightarrow{\sim} \mathcal{X}(E, -) = U_V |\cdot| = U_W \Phi |\cdot|,$$

with $|\cdot|: \mathcal{X} \longrightarrow V\text{-Cat}$ and $\Phi: V\text{-Cat} \longrightarrow W\text{-Cat}$ as in 3.1 and Section 5. We must now lift this **Set**-based isomorphism to a $W\text{-Cat}$ -based isomorphism $\theta: F \xrightarrow{\sim} \Phi |\cdot|$. For ease of computation, we may without loss of generality assume that the **Set**-based isomorphism is actually an identity; hence, FX has underlying set $\mathcal{X}(E, X)$, for all objects X in \mathcal{X} , and we must show that FX and $\Phi |X|$ have the same W -category structure. But for all $x, y \in |X| = \mathcal{X}(E, X)$ one has with 6.3(4):

$$\begin{aligned} (\Phi |X|)(x, y) &= \Phi(|X|(x, y)) \\ &= \Phi(y^* \circ x_*) \\ &= (\widetilde{F}, \widetilde{\alpha})(y^*) \circ (\widetilde{F}, \widetilde{\alpha})(x_*) \\ &= (Fy)^* \circ (Fx)_* \\ &= (FX)(x, y). \end{aligned}$$

Finally, in order to confirm θ as a 2-cell $(F, \alpha) \xrightarrow{\sim} \Gamma\Phi \cdot [|\cdot|, \gamma]$, under the assumption $\theta = 1$ we must show that the diagram

$$\begin{array}{ccccc} \Phi |S X| & \xrightarrow{\Phi_{\gamma X} = \gamma_X} & \Phi(P_V^\circ |X|) & \xrightarrow{\pi_X} & P_W^\circ \Phi X \\ \parallel & & & & \parallel \\ F S X & \xrightarrow{\alpha_X} & P_W^\circ F X & & P_W^\circ F X \end{array}$$

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commutes, for all objects X . To this end, let us first observe that α_X may be considered a W -module $FX \longrightarrow FSX$, and as such is represented as

$$\alpha_X = (\widetilde{F, \alpha})(\iota_X),$$

with $\iota_X : X \longrightarrow SX$ in $S\text{-Mod}$ (see Section 2). Now, for all $\varphi \in |SX|$ and $x \in X$, we obtain:

$$\begin{aligned} \pi_X(\gamma_X(\varphi))(x) &= \Phi(\gamma_X(\varphi)(x)) \\ &= \Phi(\varphi \circ x_*) \\ &= \Phi(\varphi^* \circ \iota_X \circ x_*) \\ &= (\widetilde{F, \alpha})(\varphi^* \circ \iota_X \circ x_*) \\ &= (\widetilde{F, \alpha})(\varphi^*) \circ (\widetilde{F, \alpha})(\iota_X) \circ (\widetilde{F, \alpha})(x_*) \\ &= (F\varphi)^* \circ \alpha_X \circ (Fx)_* \\ &= \alpha_X(\varphi)(x). \end{aligned}$$

□

Remark 6.5. While **MODMON** carries the structure of a 2-category (see 6.2(2)), the full extent of this structure is of limited interest for our purposes, since Δ maps every 2-cell to an identity morphism: see 6.3(2). Likewise, the natural 2-categorical structure of **CQuant**, given by ordering its hom-sets pointwise, is not helpful for our purposes: if $\Phi, \Psi : V \longrightarrow W$ are morphisms of commutative quantales with $\Phi(v) \leq \Psi(v)$ for all $v \in V$, the natural transformation $\theta : \Phi \longrightarrow \Psi : V\text{-Cat} \longrightarrow W\text{-Cat}$ with $\theta_X = 1_X$ (at the **Set**-level) will in general *not* give a 2-cell $\Gamma\theta \longrightarrow \Gamma\Psi$.

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